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Lagrange structure and quantization

Peter O. Kazinski, Simon L. Lyakhovich and Alexey A. Sharapov*

Department of Quantum Field Theory, Tomsk State University
Tomsk 634050, Russia
E-mail: kpo@phys.tsu.ru, sll@phys.tsu.ru, sharapov@phys.tsu.ru

Abstract: A path-integral quantization method is proposed for dynamical systems whose classical equations of motion do *not* necessarily follow from the action principle. The key new notion behind this quantization scheme is the Lagrange structure which is more general than the lagrangian formalism in the same sense as Poisson geometry is more general than the symplectic one. The Lagrange structure is shown to admit a natural BRST description which is used to construct an AKSZ-type topological sigma-model. The dynamics of this sigma-model in $d+1$ dimensions, being localized on the boundary, are proved to be equivalent to the original theory in d dimensions. As the topological sigma-model has a well defined action, it is path-integral quantized in the usual way that results in quantization of the original (not necessarily lagrangian) theory. When the original equations of motion come from the action principle, the standard BV path-integral is explicitly deduced from the proposed quantization scheme. The general quantization scheme is exemplified by several models including the ones whose classical dynamics are not variational.

Keywords: BRST Quantization, BRST Symmetry, Gauge Symmetry.

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Contents

1. Introduction

We suggest a method for path integral quantization of classical theories whose equations of motion are not necessarily variational. The key idea behind the method is that any classical dynamics can be uniformly converted into an equivalent topological field theory based on the action principle. Below in the introduction we informally comment on the main ingredients of the construction to give a preliminary impression of the quantization method we propose.

In the first place, we introduce a notion of *Lagrange structure* that generalizes the standard lagrangian formalism more or less in the same sense as the Poisson geometry generalizes the symplectic one. The Lagrange structure does not require the equations of motion to be lagrangian in usual sense, i.e. no integrating multiplier is assumed to exist bringing the equations to the variational form. The main ingredient of the Lagrange structure is the *Lagrange anchor* (denoted by V) that would be the inverse to an integrating multiplier Λ if the latter existed. The Lagrange anchor is required to satisfy a chain of compatibility conditions involving the equations of motion and the gauge generators for the gauge systems. If the anchor is invertible, these conditions will be equivalent to that providing the inverse matrix $\Lambda = V^{-1}$ to be an integrating multiplier for the equations of motion. Given the Lagrange structure, one can perform the path integral quantization of the classical theory even when the anchor is not invertible, and therefore, the dynamics do not admit any action functional.

Another explanation for the origin of the Lagrange structure is provided by the BV formalism [1, 2]. The standard BV method describes the (gauge) system in terms of field-antifield supermanifold endowed with a master action and the canonical antibracket. Due to the Jacobi identity, the corresponding antibracket is automatically compatible with the BRST differential (associated with the master action) in the sense of the Leibnitz rule. For this standard case, the Lagrange anchor appears in the theory as an odd bivector defining canonical antibracket between fields and anti-fields. Similarly, as we show, the generic Lagrange structure equips an appropriately superextended original space with a BRST differential and a (non-canonical, weak) antibracket. The BRST differential carries all the information about the original classical theory, whereas the antibracket contains the ingredients needed for quantization. To make the formalism working (i.e. sufficient for the path integral quantization), the antibracket is required to satisfy the graded Jacobi identity in a "weak" sense, i.e. up to homotopy w.r.t. the BRST differential. In this algebraic setting, all the compatibility conditions between classical equations of motion and Lagrange anchor are encoded in the graded Leibnitz rule for the BRST differential and the weak antibracket. As the antibracket can be degenerate, the BRST differential is not necessarily anti-hamiltonian vector field, in contrast to the standard lagrangian theory.

Regarding the relaxed Jacobi identity for the weak antibracket, the Lagrange structure can be considered as lagrangian counterpart of the *weak Poisson structure* studied in [3]. Various particular types of the weak Poisson brackets were studied earlier in [4−6] and recently in [7]. The deformation quantization of weak Poisson manifolds was given in ref. [3] making use of superextension to the Kontsevich formality theorem [8], which had not been proved at that moment. An exhaustive proof has been given to the superformality theorem in the recent paper [9] in connection with the deformation quantization of P_∞-structure (=weak Poisson structure).

From the pure algebraic viewpoint, the defining relations for the Lagrange structure can be regarded as structure relations for L_∞-algebra [10, 11]. This L_∞-algebra is a particular example of the homotopy analog of Schouten (= odd Poisson, Gerstenhaber) algebras studied in ref. [12] in the context of higher derived brackets. The higher (derived) brackets were also studied in various related contexts in refs. [25].

As a next step towards quantization, we work out a BRST description for the Lagrange structure of a generic (i.e. not necessarily lagrangian) classical theory. Making use of this BRST formalism, we construct a topological sigma-model along the lines of the AKSZ approach [13] and its further development.[1] [14 − 16, 6] The master equation for the sigma-model action reproduces the defining relations for the Lagrange structure of the original theory much similar as the master equation for the Poisson sigma-model reproduces the Jacobi identity for the Poisson bivector [17, 18]. By construction, the dynamics of the topological sigma-model is variational and we prove it is equivalent to the dynamics of the original classical theory. Quantizing this topological field theory one gets an arbitrary (not necessarily lagrangian) original theory quantized. If the original theory had an action, the transition amplitude for the sigma-model can be explicitly integrated out in the bulk, resulting in the standard BV answer for the path integral of the original theory, with the BV master action built in a usual way from the original action.

Notice that the trivial Lagrange anchor $V = 0$ is always admissible for any equations of motion. The trivial anchor, however, results in trivial quantization, with no quantum fluctuations appeared: The path integral for the topological sigma-model is reduced to integration only over the classical trajectories of the original theory. In this sense, the transition amplitudes in (non-lagrangian) theories with trivial Lagrange anchor are similar to the classical transition amplitudes studied by Gozzi et. al. [19] for theories based on the action principle. In the general case, the quantum fluctuations will be trivial only for those degrees of freedom which belong to the kernel of the Lagrange anchor. The quantization technique we develop is explicitly covariant and does not require any special coordinate system adjusted for separating the anchor kernel from the other degrees of freedom.[2] In this respect, the proposed path integral quantization is analogous to the deformation quantization of Poisson manifolds: the degrees of freedom from the kernel of the (regular) Poisson bivector correspond to the center of $*$-algebra upon quantization, remaining "nonquantized" in this sense.

Let us comment on the paper composition. In the next section we elaborate on the origin of the Lagrange anchor and the related structures from the viewpoint of classical dynamics. In section 3 we introduce the notion of the regular Lagrange structure and discuss its connection with strongly homotopy Schouten algebras (S_∞-algebras for short). In section 4, the BRST embedding is worked out for the Lagrange structure and the existence theorem is proved for the corresponding master equation. As a byproduct, we obtain one more geometric interpretation of the Lagrange structure as an infinitesimal deformation of certain lagrangian submanifold in the cotangent bundle over the space of

[1]There is a great number of works devoted to the topological sigma-models. We mention only those references which are the most relevant for the construction we use.

[2]Moreover, such an adjusted coordinate system does not exist if the rank of the anchor varies over the configuration space. However, this separation, whenever it is possible, allows for a simple interpretation. For example, if the degrees of freedom belonging to the kernel of the Lagrange anchor can be explicitly excluded from the classical equations of motion it is natural to consider them as auxiliary variables. The equations will become variational for the remaining degrees of freedom. The auxiliary degrees of freedom do not fluctuate in this case, while the other ones are quantized in the usual way with an action defined on the reduced space.

trajectories. In section 5, proceeding along the lines of the AKSZ procedure, we construct a topological sigma-model related to the BRST complex which have been built for the Lagrange structure in previous section. Being effectively localized on the boundary, the dynamics of this topological sigma-model in $d+1$ dimensions are proved to be equivalent to the original d-dimensional theory. Quantizing the topological sigma model we get the path integral quantization of the original (non-lagrangian) dynamics. Finally, in section 6, we consider several examples illustrating the quantization formalism proposed. As the *first example*, we show that the quantization of the usual lagrangian gauge theory, being performed by our method, is equivalent to the standard BV-quantization. Next, as the *second example*, we detail our quantization method for the general classical theory given by a set of independent (not necessarily lagrangian) equations of motion without gauge symmetry. As the *third example* we consider the systems with the equations of motion being the first-order ODEs $\dot{x}^i = h^i(x)$. For these systems, we identify purely algebraic (i.e. containing no time derivatives) Lagrange anchors with the Poisson bivectors compatible to the vector h. When the Poisson bivector is nondegenerate, the equations of motion are hamiltonian and can be derived from the first-order action functional, while in the degenerate case, these equations are not necessarily either hamiltonian, or variational. However, our method provides a natural embedding of this theory into the equivalent topological sigma-model, so the theory can be quantized through this embedding. The *fourth example* deals with the Maxwell electrodynamics formulated in terms of the strength tensor of electromagnetic field. The Maxwell equations for the strength tensor are known to be non-lagrangian and linearly dependent. We find an explicitly covariant Lagrange anchor for these equations and quantize them following our general prescription. As a result, we obtain a path integral giving precisely the same transition amplitude as usual Faddeev-Popov path integral defined in terms of the electromagnetic potentials. This exemplifies the way in which our quantization method can be made applicable to non-lagrangian field theories formulated from outset in terms of strength tensors, like the high-spin gauge fields [20].

2. Lagrange structure: a preliminary exposition

In this section, we give a down-to-earth explanation to the origin of the Lagrange anchor and related structures. It is the structures which are behind the path integral quantization of dynamical systems whose equations of motion do not necessarily follow from the action principle. More rigorous consideration of the subject is given in section 3.

In the standard lagrangian formalism, the mechanical system is specified by an action functional $S : M \to \mathbb{R}$ defined on the space of all trajectories (histories) M over the configuration space of the system. The true physical trajectories are postulated to deliver local minima to S that leads to the equations of motion of the form

$$T_i(x) \equiv \partial_i S(x) = 0\,, \tag{2.1}$$

where x^i are local coordinates on M. When all the critical points of $S(x)$ are non-degenerate (and hence isolated) one said about a non-degenerate lagrangian theory; otherwise there

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can exist continuous families of trajectories satisfying the same boundary conditions, in which case one says about a gauge invariant (or degenerate) lagrangian theory.

The equations of motion (2.1) can be understood as defined by an exact 1-form $T = dS$ on an infinite-dimensional manifold M. An immediate consequence of this interpretation is the Helmholtz criterion

$$dT = 0\,, \tag{2.2}$$

that is a necessary condition for the equations $T = 0$ to come from the action principle. Due to the Poincaré lemma, the closedness of T ensures the existence of a local action functional S_U defined on any contractible open set $U \subset M$ such that $T|_U = dS_U$. If $x_0 \in U$ is a solution to $T = 0$, one can use S_U to perform the quasi-classical (= perturbative in $\hbar$) quantization of the system "near the classical trajectory x_0".

So, the standard lagrangian formalism requires the equations of motion to satisfy two conditions: (i) they must be components of a one-form on the cotangent bundle to the space of trajectories, and (ii) this one-form has to be closed, i.e satisfying the Helmholtz condition (2.2). Of course, there are many physically interesting systems whose equations of motion do not satisfy even the first condition, not to mention the second one. The first natural step towards generalizing the lagrangian formalism is to replace the cotangent bundle T^*M, where the equations (2.1) take the values, by an arbitrary vector bundle $\mathcal{E} \to M$, that we term a *dynamics bundle*. The space of true physical trajectories is then identified with zero locus of some section in the dynamics bundle: $T \in \Gamma(\mathcal{E})$. (Relaxing the Helmholtz condition will be the next step, addressed after relation (2.7)). If e^a is a local frame of sections of $\mathcal{E}$ over a trivializing coordinate chart $U \in M$, so that $T = T_a(x)e^a$, then instead of (2.1) we get the equations of motion in the form:

$$T_a(x) = 0\,. \tag{2.3}$$

Notice that we do not assume that $\dim T^*M = \dim \mathcal{E}$, so the "number" of equations is allowed to be less or greater than $\dim M$. (Of course, in the infinite dimensional context, the notion of dimension needs clarification. An appropriate definition can be done, for example, in the case of local theories, i.e. when (2.3) is a system of PDE's.)

The question arises what might be an analogue for the integrability condition (2.2) that can ensure the existence of a local action for the equations (2.3). To answer this question, let us first consider the case when $\dim \mathcal{E} = \dim T^*M$. In this case, the answer is given by existence of a vector bundle isomorphism $\Lambda : \mathcal{E} \to T^*M$ such that $T' = \Lambda(T)$ is a closed 1-form. In terms of local coordinates this reads

$$T'_i(x) = \Lambda^a_i(x) T_a(x)\,, \tag{2.4}$$

where $\Lambda^a_i(x)$ is a non-degenerate matrix. The equations $T' = 0$ are obviously equivalent to eqs.(2.3) in the sense that both have the same solutions. Checking the closedness condition for the 1-form T' leads to the following relations:

$$dT' = 0 \quad \Leftrightarrow \quad dT_a = \Lambda^b C^d_{ab} T_d + G_{ab}\Lambda^b\,. \tag{2.5}$$

Here we consider T_a and $\Lambda^a = \Lambda^a_i dx^i$ as a collection of 0- and 1-forms defined on a coordinate chart U and labelled by index a. The structure functions G_{ab} and C^c_{ab} are, respectively, symmetric and antisymmetric in indices a, b. In particular, the functions C^c_{ab} enter to the Maurer-Cartan equation for the basis 1-forms Λ^a:

$$d\Lambda^a = C^a_{bc}\Lambda^b \wedge \Lambda^c \,. \tag{2.6}$$

It is the question of finding the "integrating multiplier" Λ or investigating obstructions to its existence that the inverse problem of variational calculus deals with. As soon as Λ is known, one can define a local action $S(x)$ such that $dS = T_a\Lambda^a \equiv T'$. Having the local action at hands, one can perform a quasi-classical path integral quantization in a vicinity of any classical solution.

Since $\Lambda : \mathcal{E} \to T^*M$ is an isomorphism of vector bundles, there is the inverse map $V = \Lambda^{-1} : \mathcal{E}^* \to TM$ defining (and defined by) a section $V = V^i_a(x)e^a \otimes \partial_i \in \Gamma(\mathcal{E} \otimes TM)$. The integrability condition (2.5) is then equivalent to the following relation in terms of V_a and T_a:

$$V^i_a\partial_i T_b - V^i_b\partial_i T_a = C^d_{ab}T_d \,. \tag{2.7}$$

Now one may forget about Λ, taking the last relation as a definition of the integrability condition, valid for an arbitrary vector bundle $\mathcal{E} \to M$. In doing so, one has no need to require the homomorphism $V : \mathcal{E} \to TM$ to be of constant rank over M. It is the map V subject to relations (2.7) which we shall call the *Lagrange anchor*. The general and precise definition of the Lagrange anchor is given in the next section. When the equations of motion are defined by a 1-form on M, i.e $\mathcal{E} = TM$, the relation (2.7) still remains much less restrictive for the dynamics than the Helmholtz condition (2.5), as the anchor $V^i_j(x)$ satisfying (2.7), is not required to be invertible. The usual lagrangian dynamics (2.1) corresponds to the special case where $\mathcal{E} = TM$ and $V^j_i = \delta^j_i = \Lambda^j_i$.

For the general Lagrange anchor (2.7), no integrating multiplier $\Lambda = V^{-1}$ is required to exist. So, in general, existence of the Lagrange anchor (2.7) does not mean existence of a local action S_U in the vicinity U of given solution x_0. Nonetheless, if x_0 is a *regular* point, in the sense that the ranks of matrices (V^i_a) and $(\partial_i T_a)$ are constant over U, the following statement holds true:

Proposition 1. *Given a pair of sections (T, V) satisfying (2.7), then for any regular solution $x_0 \in M$ of (2.3) one can find a coordinate system $(y^1, \ldots, y^m, z^1, \ldots, z^k)$ centered at x_0 together with a set of local functions $S(y)$, $E^1(y), \ldots, E^k(y)$ such that equations $T_a(y, z) = 0$ are equivalent to*

$$\frac{\partial S(y)}{\partial y^I} = 0 \,, \qquad z^J = E^J(y) \,,$$

where $k = \mathrm{rank}(\partial_i T_a(x_0)) - \mathrm{rank}(G_{ab}(x_0))$ and $G_{ab} = V^i_a\partial_i T_b$.

We prove this proposition in section 4.6. It is natural to call the function $S(y)$, depending on a part of degrees of freedom, a *partial action*. On the surface defined by the equations for

z's[3], the dynamics become lagrangian for the other degrees of freedom, more or less in the same sense as the hamiltonian dynamics become symplectic upon reduction to a symplectic leaf of a regular Poisson structure. It should be noted that the path integral quantization method developed in this paper does not require any special coordinate system. Moreover, the method is insensitive to the rank of the matrix $V_a^i(x)$ and remains applicable to the cases where no (partial) action can exist.

Let Σ denote the set of all solutions, i.e. $\Sigma = \{p \in M | T_a(p) = 0\}$. Equations of motion (2.3) are called *dependent* if there is a vector bundle $\mathcal{E}_1 \to M$ and a bundle homomorphism $Z : \mathcal{E} \to \mathcal{E}_1$ such that $T \in \ker Z$ and $\mathrm{rank}(Z|_\Sigma) \neq 0$. In terms of local coordinates this means

$$Z_A^a T_a \equiv 0\,, \tag{2.8}$$

where the section $Z = Z_A^a e_a \otimes e^A \in \Gamma(\mathcal{E}^* \otimes \mathcal{E}_1)$ does not vanish on Σ identically.

Equations of motion (2.3) are said to be *gauge invariant* if there exists a vector bundle $\mathcal{E}_{-1} \to M$ together with a bundle homomorphism $R : \mathcal{E}_{-1} \to TM$ such that the corresponding section $R = R_\alpha^i e^\alpha \otimes \partial_i \in \Gamma(\mathcal{E}_{-1}^* \otimes TM)$ does not vanish on Σ identically and

$$R_\alpha^i \partial_i T_a|_\Sigma = 0\,. \tag{2.9}$$

For an ordinary gauge theory with action S and gauge symmetry generators $R_\alpha^i \partial_i$, rels. (2.8), (2.9) take the form

$$R_\alpha^i \partial_i S = 0\,, \qquad R_\alpha^i \partial_i \partial_j S|_{dS=0} = 0\,. \tag{2.10}$$

As is seen, in the ordinary lagrangian gauge theory, the role of R's is two-fold: the same R's generate the gauge symmetries of the equations of motion (2.9) and describe the functional dependence (the Noether identities) between them (2.8). If the equations follow from the action principle, the gauge symmetry means dependence of equations of motion, and vice versa. In the general (non-lagrangian) case, the generators R's and Z's may be completely independent from each other. In particular, it is possible to have dependent but not gauge invariant equations of motion and vice versa.

Rels. (2.8), (2.9) considered independently from (2.7) define a gauge algebra structure irrespectively to existence of the lagrangian or hamiltonian formalism. The BRST imbedding for such a generic gauge algebra was systematically described in ref. [3] along the usual lines of the BRST theory[4] [2]. Examining the compatibility conditions between the defining relations for the gauge algebra (2.8, 2.9) and the Lagrange structure (2.7), one can find rich algebraic and geometric structures that are systematically studied in the following sections. These are the structures which provide the possibility to (path-integral) quantize the system even if its classical dynamics does not admit action principle.

[3] In fact, these equations are not required to be explicitly solved w.r.t. z's, just a unique existence is required for the solution with appropriate initial data.

[4] Also in ref. [3], the stress was made on consistent combining of the gauge algebra relations (2.8, 2.9) and the (weak) Poisson structure. That was aimed at deformation quantization of the generic gauge systems (not necessarily having Poisson structure on M) through constructing a star-product which is associative only for the on-shell gauge invariants, not for all functions on M. Examples of the non-lagrangian and/or non-hamiltonian models where the BRST description is an efficient tool for studying the classical dynamics, can be found, e.g., in refs. [21, 22].

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3. Regular Lagrange structure and S_∞-algebra

To summarize the previous discussion in a more formal way, a classical system is specified by a vector bundle $\mathcal{E} \to M$ over the space of trajectories M, and a section $T \in \Gamma(\mathcal{E})$ playing the role of equations of motion. The space of true trajectories of the system is then identified with zero locus Σ of T: $\Sigma = \{x \in M | T(x) = 0\}$. In what follows we refer to Σ as a *shell*.

3.1 Lagrange structure

By a *Lagrange structure* for a classical system $(\mathcal{E}, T)$ we understand $\mathbb{R}$-linear map $d_\mathcal{E} : \Gamma(\wedge^n \mathcal{E}) \to \Gamma(\wedge^{n+1}\mathcal{E})$ obeying conditions:

(i) $d_\mathcal{E} T = 0\,,$

(ii) $d_\mathcal{E}$ is a derivation of degree 1, i.e.

$$d_\mathcal{E}(A \wedge B) = d_\mathcal{E}A \wedge B + (-1)^n A \wedge d_\mathcal{E} B\,, \qquad \forall A \in \Gamma(\wedge^n\mathcal{E}),\ \forall B \in \Gamma(\wedge^\bullet \mathcal{E})\,.$$

Here we identify $\Gamma(\wedge^0\mathcal{E})$ with $C^\infty(M)$.

Due to the Leibnitz identity (ii), in each trivializing chart $U \subset M$ the operator $d_\mathcal{E}$ is completely specified by its action on coordinate functions x^i and basis sections e^a of $\mathcal{E}|_U$:

$$d_\mathcal{E} x^i = V_a^i(x) e^a\,, \qquad d_\mathcal{E} e^a = -\frac{1}{2} C^a_{bc}(x) e^b \wedge e^c\,. \tag{3.1}$$

Applying $d_\mathcal{E}$ to the section $T = T_a e^a$, one can see that the property (i) reproduces the integrability condition (2.7):

$$0 = d_\mathcal{E} T = \frac{1}{2}(V_a^i \partial_i T_b - V_b^i \partial_i T_a - C^c_{ab} T_c) e^a \wedge e^b\,.$$

The first relation from (3.1) means also that $d_\mathcal{E}$ defines a bundle homomorphism $V : \mathcal{E}^* \to TM$.

In the particular case where $d_\mathcal{E}{}^2 = 0$, T is nothing but a closed 1-$\mathcal{E}$-form associated to the Lie algebroid with the anchor V. Although for the general Lagrange structure (i)-(ii), the differential $d_\mathcal{E}$ is not required to be nilpotent, we call V the *Lagrange anchor*. For the Lagrange anchor, the requirement of identical nilpotency is replaced by a relaxed condition $d_\mathcal{E}{}^2 T = 0$ following from (i). This weaker requirement can have further off-shell consequences that are derived in the next section under certain regularity conditions on T.

3.2 Regularity conditions

Let $(\mathcal{E}, T, d_\mathcal{E})$ be a Lagrange structure with shell Σ. The Lagrange structure is said to be *regular of type* (m, n) if $\Sigma \neq \emptyset$ and there exists a finite chain of vector bundles $\mathcal{E}_k \to M$ together with M-bundle homomorphisms

$$0 \to \mathcal{E}_{-m} \to \cdots \to \mathcal{E}_{-1} \xrightarrow{R} TM \xrightarrow{J} \mathcal{E} \xrightarrow{Z} \mathcal{E}_1 \to \cdots \to \mathcal{E}_n \to 0\,, \tag{3.2}$$

such that

(a) the map J is defined by section $\nabla T \in \Gamma(T^*M \otimes \mathcal{E})$, where ∇ is any connection on $\mathcal{E}$;

(b) there is a neighbourhood $U \subset M$ of Σ such that all the homomorphisms (3.2) have constant ranks over U;

(c) upon restriction to Σ, the chain (3.2) makes an exact sequence.

Several remarks are in order concerning this definition.

Remark 1. The regularity condition ensures that $\Sigma \subset M$ is a smooth submanifold.

Remark 2. When exist, the homomorphisms (3.2) are not unique off shell. Thinking of these homomorphisms as sections of the corresponding vector bundles,

$$R = R^i_\alpha e^\alpha \otimes \partial_i\,, \qquad J = \nabla_i T_a dx^i \otimes e^a\,, \qquad Z = Z^a_A e_a \otimes e^A\,, \qquad \ldots\,, \tag{3.3}$$

one can add to them any sections vanishing on Σ, leaving the properties (a)-(c) unaffected. In particular, making a shift

$$Z \to Z + T_a W^{ab}_A e^A \otimes e_b\,, \tag{3.4}$$

if necessary, we can always choose Z in such a way that $T \in \ker Z$, cf. (2.8).

Remark 3. In the definition above one can pass from the chain (3.2) to the transpose one by replacing each vector bundle with its dual and inverting all the arrows. The transpose chain meets the same conditions (a)–(c) as the original one.

Remark 4. The condition (c) means that R's and Z's, defining the chain links in (3.2), are to be understood as the generators of gauge symmetry (2.9) and the generators of Noether identities (2.8). Having in mind this interpretation, we term $\mathcal{E}_{-1}$ and $\mathcal{E}_1$ the *gauge algebra bundle* and the *Noether identity bundle*, respectively. Accordingly, $\mathcal{E}$ is referred to as the *dynamics bundle* .

In this paper we deal mostly with quantization of regular $(1,1)$-type Lagrange structures associated to the four-term sequences

$$0 \to \mathcal{F} \xrightarrow{R} TM \xrightarrow{J} \mathcal{E} \xrightarrow{Z} \mathcal{G} \to 0\,. \tag{3.5}$$

In other words, we consider a set of gauge invariant and linearly dependent equations of motion (2.3), (2.8), (2.9) with the generators of gauge symmetry R_α and Noether identities Z_A chosen in a linearly independent way. In the ordinary lagrangian gauge theory the dynamics bundle coincides with the cotangent bundle ($\mathcal{E} = T^*M$), the Noether identity bundle coincides with the gauge algebra bundle ($\mathcal{E}_{-1} = \mathcal{E}_1$) and the generators of gauge symmetry coincide with the generators of Noether identities ($R = Z$). For the general system of type $(1,1)$, any of these coincidences should not necessarily occur, e.g.: the gauge algebra bundle $\mathcal{E}_{-1}$ and the bundle of Noether identities $\mathcal{E}_1$ can be different even by dimension. In section 6.4 we give an example of quantizing the dynamical system of type (0,1) which is not lagrangian, although it has a nontrivial Lagrange anchor. This means the theory has dependent equations of motion having no gauge symmetry.

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3.3 Completeness

A regular Lagrange structure $(\mathcal{E}, T, d_{\mathcal{E}})$ is called *complete* if

$$TM = \operatorname{Im} V \cup \operatorname{Im} R\,, \tag{3.6}$$

where $V : \mathcal{E}^* \to TM$ is the Lagrange anchor corresponding to $d_{\mathcal{E}}$, and R is determined by (3.2). In other words, the completeness means that the tangent bundle is spanned by the Lagrange anchor and the gauge symmetry generators. It is easy to find that the number m of lagrangian equations in proposition 1 is equal to the rank of the matrix (R^i_α, V^i_a). Hence, for a complete Lagrange structure, all the equations of motion turn out to be (locally) lagrangian. In this paper we consider regular Lagrange structures that are not necessarily complete.

3.4 Physical observables

Given a regular Lagrange structure $(\mathcal{E}, T, d_{\mathcal{E}})$ with the shell Σ, we say that $f \in C^\infty(M)$ is a *trivial* function if it vanishes on shell, i.e. $f|_\Sigma = 0$. The subspace of trivial functions is denoted by $C^\infty(M)^{\text{triv}}$. It follows from the regularity conditions that

$$f \in C^\infty(M)^{\text{triv}} \quad \Leftrightarrow \quad f = K^a T_a\,,$$

for some $K \in \Gamma(\mathcal{E}^*)$. Function $f \in C^\infty(M)$ is said to be *invariant* if for any section $\varepsilon = \varepsilon^\alpha e_\alpha \in \Gamma(\mathcal{E}_{-1})$ there exist a section $F \in \Gamma(\mathcal{E}^* \otimes \mathcal{E}^*_{-1})$ such that

$$\varepsilon^\alpha R^i_\alpha \partial_i f = \varepsilon^\alpha F^a_\alpha T_a \in C^\infty(M)^{\text{triv}}\,, \tag{3.7}$$

Here $\{R_\alpha\}$ is an (over)complete basis of gauge generators (2.9). Again, in view of the regularity conditions one can rewrite (3.7) in a more compact way: $R^i_\alpha \partial_i f|_\Sigma = 0$. The subspace of invariant functions is denoted by $C^\infty(M)^{\text{inv}}$. In view of condition (2.9) the trivial functions are automatically invariant, so we can write

$$C^\infty(M)^{\text{triv}} \subset C^\infty(M)^{\text{inv}} \subset C^\infty(M)\,.$$

Two invariant functions $f_1, f_2 \in C^\infty(M)^{\text{inv}}$ are considered as equivalent if they coincide on shell,

$$f_1 \sim f_2 \;\Leftrightarrow\; f_1 - f_2 = K^a T_a \in C^\infty(M)^{\text{triv}}\,. \tag{3.8}$$

The space of physical observables $\mathcal{P}$ is now defined as the quotient of the space of invariant functions by the space of trivial ones,

$$\mathcal{P} = \frac{C^\infty(M)^{\text{inv}}}{C^\infty(M)^{\text{triv}}} \tag{3.9}$$

Let us recall that M is an infinite dimensional space of trajectories, and $T_a = 0$ are the differential equations whose solutions are parametrized by initial data. The initial data transformed into each other by the gauge symmetry transformations are considered as equivalent. The space of inequivalent initial data can then be understood as a physical phase space M_{phys}. In the case where M_{phys} happen to be a smooth Hausdorf manifold we have an equivalent definition of $\mathcal{P}$ as the space of smooth functions on M_{phys}, i.e. $\mathcal{P} = C^\infty(M_{\text{phys}})$.

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3.5 S_∞-algebras

An S_∞-algebra (S for Schouten) is a $\mathbb{Z}_2$-graded, supercommutative and associative algebra A endowed with a sequence of odd linear maps $S_n : A^{\otimes n} \to A$ such that

(i) $S_n(\dots, a_k, a_{k+1}, \dots) = (-1)^{\epsilon(a_k)\epsilon(a_{k+1})} S_n(\dots, a_{k+1}, a_k, \dots)$,
$\epsilon(a)$ being the parity of a homogeneous element $a \in A$.

(ii) $a \mapsto S_n(a_1, \dots, a_{n-1}, a)$ is a derivation of A of the parity $1 + \sum_{k=1}^{n-1} \epsilon(a_k)$ (mod 2).

(iii) For all $n \geq 0$,

$$\sum_{k+l=n} \sum_{(k,l)-\text{shufle}} (-1)^\epsilon S_{l+1}(S_k(a_{\sigma(1)}, \dots, a_{\sigma(k)}), a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)}) = 0\,,$$

where $(-1)^\epsilon$ is the natural sign prescribed by the sign rule for a permutation of homogeneous elements $a_1, \dots, a_n \in A$.

Recall that a (k,l)-shuffle is a permutation of indices $1, 2, \dots, k+l$ satisfying $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

When $S_0 = 0$ we say about a *flat* S_∞-algebra. In this case $S_1 : A \to A$ is a nilpotent differential, and S_2 induces an odd Poisson structure on corresponding cohomology. An odd Poisson algebra can thus be regarded as S_∞-algebra with bracket $S_2 : A \otimes A \to A$ and all other $S_k = 0$. In fact, properties (i) and (iii) characterize L_∞-algebras. See [12] for recent discussion of S_∞-algebras.

It turns out that any regular Lagrange structure of type (m,n) gives rise to an S_∞-algebra structure on the supercommutative algebra of sections

$$A = \Gamma\Big(\wedge^\bullet \mathcal{E} \otimes \bigotimes_{k=1}^{m} S^\bullet(\Pi^k \mathcal{E}_{-k}) \otimes \bigotimes_{l=1}^{n} S^\bullet(\Pi^{l+1}\mathcal{E}_l) \Big). \tag{3.10}$$

Here $S^\bullet$ stands for symmetric tensor powers (in the $\mathbb{Z}_2$-graded sense) and Π denotes the parity reversion operation, i.e. $\Pi\mathcal{E}$ is a vector bundle over M whose fibers are odd linear spaces. By definition, $\Pi^2 = \mathrm{id}$ and $S^\bullet(\Pi\mathcal{E}) = \wedge^\bullet \mathcal{E}$.

In the next section, applying the machinery of the BRST theory, we give an explicit description for S_∞-algebras associated with $(1,1)$-type Lagrange structures. Extension to the (m,n)-type Lagrange structures is straightforward.

4. BRST imbedding

4.1 Ambient Poisson supermanifold

Let $(\mathcal{E}, T, d_{\mathcal{E}})$ be a regular Lagrange structure of type $(1,1)$ corresponding to the four-term sequence (3.5). Following the general line of ideas of BRST theory, we have to realize the original space of trajectories M as a body of an appropriate $\mathbb{Z}$-graded supermanifold $\mathcal{N}$. Let us choose $\mathcal{N}$ to be the total space of the vector bundle

$$\mathcal{N} = \Pi(\mathcal{F} \oplus \mathcal{F}^*) \oplus T^*M \oplus \Pi(\mathcal{E} \oplus \mathcal{E}^*) \oplus (\mathcal{G} \oplus \mathcal{G}^*)\,, \tag{4.1}$$

base and fibers	M	T^*M	$\mathcal{F}$	$\mathcal{F}^*$	$\mathcal{E}$	$\mathcal{E}^*$	$\mathcal{G}$	$\mathcal{G}^*$
local coordinates	x^i	$\bar{x}_j$	c^α	$\bar{c}_\beta$	η_a	$\bar{\eta}^b$	ξ_A	$\bar{\xi}^B$
ϵ = Grassman's parity	0	0	1	1	1	1	0	0
gh = ghost number	0	0	1	-1	-1	1	-2	2
Deg = momentum degree	0	1	0	1	0	1	0	1
deg = resolution degree	0	1	0	2	1	0	2	0

Table 1: The gradings.

where $\mathcal{F},\mathcal{E}$ and $\mathcal{G}$ are the bundles of gauge algebra, dynamical equations and the Noether identities respectively (3.5), see remark 4 of section 3.2. The base M is canonically imbedded into $\mathcal{N}$ as zero section. Besides the Grassman parity, the fibers of each direct summand in (4.1) are endowed with an additional $\mathbb{Z}$-grading, called the *ghost number*. For simplicity, to avoid cumbersome sign factors, we assume the base M to be an ordinary (even) manifold, that corresponds to the case of mechanical systems without fermionic degrees of freedom. Then the Grassman parity of the fibers is correlated to $\mathbb{Z}$-grading in a simple way: the even coordinates have even ghost numbers, while the odd coordinates have odd ghost numbers. We also equip $\mathcal{N}$ with a pair of auxiliary $\mathbb{N}$-gradings called the *momentum-* and *resolution degrees* (m- and r-degrees, for short) that will be used later for proving the existence theorem for the BRST charge. The information about the gradings of local coordinates is arranged in table 1.

Splitting all the local coordinates into the "position coordinates" $\varphi^I = (x^i, c^\alpha, \eta_a, \xi_A)$ and "momenta" $\bar{\varphi}_J = (\bar{x}_i, \bar{c}_\alpha, \bar{\eta}^a, \bar{\xi}^A)$, we can write

$$\begin{array}{ll} \mathrm{gh}(\bar{\varphi}_I) = -\mathrm{gh}(\varphi^I)\,, & \epsilon(\bar{\varphi}_I) = \epsilon(\varphi^I)\,, \\ \mathrm{Deg}(\bar{\varphi}_I) = 1\,, & \mathrm{Deg}(\varphi^I) = 0\,. \end{array} \tag{4.2}$$

Though the submanifold $T^*M \subset \mathcal{N}$ is an ordinary manifold, we do not include the fibers of T^*M into the body of $\mathbb{Z}\times\mathbb{Z}_2$-graded manifold N and treat $\bar{x}$'s as formal variables.

Fixing a linear connection $\nabla = \nabla_{\mathcal{F}} \oplus \nabla_{\mathcal{E}} \oplus \nabla_{\mathcal{G}}$ on $\mathcal{F}\oplus\mathcal{E}\oplus\mathcal{G}$, we endow $\mathcal{N}$ with the exact symplectic structure $\omega = d\Lambda$, where

$$\begin{aligned} \Lambda &= \bar{x}_i dx^i + \bar{c}_\alpha \nabla c^\alpha + \bar{\eta}^a \nabla \eta_a + \bar{\xi}^A \nabla \xi_A\,, \\ \nabla c^\alpha &= dc^\alpha + dx^i \Gamma^\alpha_{i\beta} c^\beta\,, \end{aligned} \tag{4.3}$$

and similar expressions are assumed for covariant differentials of η's and ξ's. The corresponding Poisson brackets of local coordinates read

$$\begin{aligned} \{\bar{\eta}^b, \eta_a\} &= \delta^b_a\,, \qquad \{\bar{x}_i, \eta_a\} = \Gamma^b_{ia}\eta_b\,, \qquad \{\bar{x}_i, \bar{\eta}^b\} = -\Gamma^b_{ia}\bar{\eta}^a\,, \\ \{\bar{c}_\alpha, c^\beta\} &= \delta^\beta_\alpha\,, \qquad \{\bar{x}_i, c^\alpha\} = \Gamma^\alpha_{i\beta}c^\beta\,, \qquad \{\bar{x}_i, \bar{c}_\beta\} = -\Gamma^\alpha_{i\beta}\bar{c}_\alpha\,, \\ \{\bar{\xi}^A, \xi_B\} &= \delta^A_B\,, \qquad \{\bar{x}_i, \xi_A\} = \Gamma^B_{iA}\xi_B\,, \qquad \{\bar{x}_i, \bar{\xi}^A\} = -\Gamma^A_{iB}\bar{\xi}^B\,, \\ \{\bar{x}_i, x^j\} &= \delta^j_i\,, \qquad \{\bar{x}_i, \bar{x}_j\} = R^b_{ija}\bar{\eta}^a\eta_b + R^\beta_{ij\alpha}c^\alpha\bar{c}_\beta + R^B_{ijA}\bar{\xi}^A\xi_B\,, \end{aligned} \tag{4.4}$$

and the other brackets vanish. The structure functions determining the Poisson brackets of $\bar{x}_i$ and $\bar{x}_j$ are just components of the curvature tensor of ∇.

Clearly, the equations $\bar{\varphi}_I = 0$ define the lagrangian submanifold

$$\mathcal{L} = \Pi(\mathcal{F} \oplus \mathcal{E}) \oplus \mathcal{G} \subset \mathcal{N}\,, \tag{4.5}$$

and the supercommutative algebra of functions $C^\infty(\mathcal{L})$ is naturally isomorphic to the algebra (3.10) with $m = n = 1$.

4.2 BRST charge

It turns out that all the ingredients of the Lagrange structure can be naturally interpreted as coefficients of expansion in the fiber coordinates of a single function $\Omega \in C^\infty(\mathcal{N})$, called the BRST charge, such that

$$\mathrm{gh}(\Omega) = 1\,, \qquad \epsilon(\Omega) = 1\,, \qquad \mathrm{Deg}(\Omega) \geq 1\,. \tag{4.6}$$

The relations defining the Lagrange structure are generated by and are equivalent to the master equation

$$\{\Omega, \Omega\} = 0\,. \tag{4.7}$$

To get the desired interpretation let us first expand Ω in the powers of momenta $\bar{\varphi}_I$:

$$\Omega = \sum_{n=1}^{\infty} \Omega_n\,, \qquad \mathrm{Deg}(\Omega_n) = n\,. \tag{4.8}$$

Substituting this expansion into the master equation (4.7) and considering it in the first three orders in the momenta, we get the following relations for Ω_1, Ω_2:

$$\{\Omega_1, \Omega_1\} = 0\,, \qquad \{\Omega_1, \Omega_2\} = 0\,, \qquad \{\Omega_2, \Omega_2\} = 2\{\Omega_1, \Omega_3\}\,. \tag{4.9}$$

The first term $\Omega_1 = \Omega^I(\varphi)\bar{\varphi}_I$ gives rise to the odd, nilpotent vector field on $\mathcal{L}$,

$$Q \equiv \Omega^I(\varphi)\frac{\partial}{\partial \varphi^I} = T_a \frac{\partial}{\partial \eta_a} + c^\alpha R^i_\alpha \frac{\partial}{\partial x^i} + \eta_a Z^a_A \frac{\partial}{\partial \xi_A} + \cdots\,, \tag{4.10}$$

carrying all the information about the classical system $(\mathcal{E}, T)$ itself. Evaluating the nilpotency condition $Q^2 = 0$ at the lowest order in r-degree (see table 1), one immediately recovers rels. (2.8, 2.9) characterizing (2.3) as a set of gauge invariant and linearly dependent equations of motion, with R and Z being the generators of gauge transformations and Noether identities, respectively.[5]

Notice that the odd vector fields with zero square are known as *homological.* These are essentially equivalent to the notion of L_∞-*algebras* or *strong(ly) homotopy Lie algebras* [10, 11]. In physical literature, the homological vector fields usually appear as *BRST-differentials* associated either with the BV master action [1, 2] or hamiltonian BFV-BRST charge [23, 2].

[5]As it has been already mentioned, these generators, coinciding in the standard lagrangian case, can be different in general, even by number.

The Poisson action of Ω_1 makes $C^\infty(\mathcal{N})$ a cochain complex with the nilpotent differential

$$\mathbb{D}A = \{\Omega_1, A\}\,, \qquad \forall A \in C^\infty(\mathcal{N})\,. \tag{4.11}$$

Let us denote by $\mathcal{H}(\mathbb{D}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n(\mathbb{D})$ the corresponding cohomology group graded by m-degree.

The Lagrange anchor $V : \mathcal{E}^* \to TM$, associated to the Lagrange structure for the classical system (4.10) is contained in the next term

$$\Omega_2 = \Omega^{IJ}(\varphi)\bar{\varphi}_I\bar{\varphi}_J = \bar{\eta}^a V_a^i \bar{x}_i + \cdots\,. \tag{4.12}$$

Rels. (4.9) characterize Ω_2 as a weak anti-Poisson structure on $\mathcal{L}$ (4.5), i.e. Ω_1-invariant ($=\mathbb{D}$-closed) odd bivector satisfying the Jacobi identity up to homotopy. The corresponding weak antibracket reads

$$(a,b) \equiv \frac{1}{2}\{\{\Omega_2, a\}, b\}\,, \qquad a,b \in C^\infty(\mathcal{L})\,. \tag{4.13}$$

Examining the Jacobi identity for these brackets one finds

$$\begin{aligned}(a,(b,c)) + (-1)^{\epsilon(b)\epsilon(c)}((a,c),b) + (-1)^{\epsilon(a)(\epsilon(b)+\epsilon(c))}((b,c),a) &= \\ = -S_3(\mathbb{D}a,b,c) - (-1)^{\epsilon(a)\epsilon(b)}S_3(a,\mathbb{D}b,c) &- \\ - (-1)^{(\epsilon(a)+\epsilon(b))\epsilon(c)}S_3(a,b,\mathbb{D}c) - \mathbb{D}S_3(a,b,c)\,.\end{aligned} \tag{4.14}$$

where we have introduced the following notation:

$$S_n(a_1,a_2,\ldots,a_n) \equiv \frac{1}{n!}\{\ldots\{\Omega_n, a_1\}a_2\},\ldots,a_n\}\,, \qquad a_k \in C^\infty(\mathcal{L})\,. \tag{4.15}$$

In particular,

$$\begin{aligned}S_0 \equiv 0\,, \quad S_1(a) = \mathbb{D}a = \Omega^I\partial_I a\,, \quad S_2(a,b) = (a,b) = (-1)^{\epsilon(a)\epsilon(I)}\Omega^{IJ}\partial_J a\partial_I b\,,\\ S_3(a,b,c) = (-1)^{\epsilon(a)(\epsilon(I)+\epsilon(J))+\epsilon(b)\epsilon(I)}\Omega_3^{IJK}\partial_K a\partial_J b\partial_I c\,.\end{aligned} \tag{4.16}$$

As is seen, the weak antibracket (4.13) induces the genuine antibracket on the cohomology group $\mathcal{H}^0(\mathbb{D})$.

As was noticed in [12], rel. (4.15) defines a flat S_∞-structure on the supercommutative algebra $C^\infty(\mathcal{L})$: By definition, each S_n is a graded-symmetric multi-differentiation of $C^\infty(\mathcal{L})$ and the generalized Jacobi identities for the collection of maps $\{S_n\}$ follow from the master equation (4.7) for the BRST charge Ω.

4.3 The unique existence of the BRST charge

In our treatment of the Lagrange structures the BRST charge Ω is not given *a priory* - it arises as a solution to the master equation (4.7) with prescribed “boundary conditions”. By the boundary conditions we mean the starting row of expansion of the BRST charge according to r-degree:

$$\Omega = T_a\bar{\eta}^a + c^\alpha R_\alpha^i\bar{x}_i + \eta_a Z_A^a\bar{\xi}^A + \bar{\eta}^a V_a^i\bar{x}_i + \cdots\,, \tag{4.17}$$

where the dots stand for the terms more than quadratic in the fiber coordinates and/or r-degree > 1. The structure functions T's, R's, Z's and V's are naturally identified with the equations of motions, gauge symmetry generators, generators of Noether identities and the Lagrange anchor, respectively.

Given the boundary conditions (4.17), the existence of a solution to the master equations (4.7) can be proved by the standard tools of homological perturbation theory [2], [24]. Let us sketch this proof. Consider the following expansion of the BRST charge according to the r-degree:

$$\Omega = \sum_{n=0}^{\infty} \Omega^{(n)} , \qquad \deg\left(\Omega^{(n)}\right) = n . \tag{4.18}$$

In particular, the general expressions for the first two terms read

$$\Omega^{(0)} = T_a \bar{\eta}^a , \qquad \Omega^{(1)} = c^\alpha R^i_\alpha \bar{x}_i + \eta_a Z^a_A \bar{\xi}^A + \bar{\eta}^a V^i_a \bar{x}_i + \bar{\eta}^a \eta_b U^b_{\alpha a} c^\alpha + \bar{\eta}^a \bar{\eta}^b W^d_{ab} \eta_d . \tag{4.19}$$

Substituting (4.18) into the master equation (4.7) gives a chain of equations of the form:

$$\delta\Omega^{(n+1)} = K_n(\Omega^{(0)}, \dots, \Omega^{(n)}) , \qquad n = 0, 1, 2, \dots , \tag{4.20}$$

where

$$\delta = T_a \frac{\partial}{\partial \eta_a} + \eta_a Z^a_A \frac{\partial}{\partial \xi_A} + \bar{x}_i R^i_\alpha \frac{\partial}{\partial \bar{c}_\alpha} + \bar{\eta}^a \nabla_i T_a \frac{\partial}{\partial \bar{x}_i} + \bar{\eta}^a \eta_b U^b_{\alpha a} \frac{\partial}{\partial \bar{c}_\alpha} \tag{4.21}$$

is a nilpotent operator decreasing the r-degree by one,

$$\delta^2 = 0 , \qquad \deg(\delta) = -1 , \tag{4.22}$$

and K_n involves the brackets of the Ω's of lower order. The nilpotency condition (4.22) is due to the following relations (cf. eqs. (2.8,2.9)):

$$Z^a_A T_a = 0 , \qquad R^i_\alpha \nabla_i T_a = U^b_{\alpha a} T_b . \tag{4.23}$$

Let $\mathcal{H}(\delta) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(\delta)$ denote the corresponding cohomology group graded by r-degree. It is not hard to see that the regularity conditions of section 3.2, which we assume satisfied for the classical system under consideration, provide acyclicity of δ in strictly positive r-degrees:

$$\mathcal{H}_n(\delta) = 0 \quad \text{for} \quad n > 0 . \tag{4.24}$$

The proof of the last fact is quite standard (see e.g. [2]) and we leave it to the reader.

On the other hand, expanding the Jacobi identity $\{\{\Omega, \Omega\}, \Omega\} = 0$ in terms of r-degree, one can deduce that the r.h.s. of the $(n+1)$-th equation (4.20) is δ-closed provided all the previous equations are satisfied. Therefore, the only equation to check to ensure solvability of (4.20) is

$$\delta\Omega^{(1)} = K_0(\Omega^{(0)}) , \tag{4.25}$$

where $K_0(\Omega^{(0)}) \equiv \{\Omega^{(0)}, \Omega^{(0)}\} = 0$. Substituting the explicit expressions (4.19), (4.21) into (4.25), we reproduce the basic relations (4.23) as well as the integrability condition (2.7) with $C^d_{ab} = W^d_{ab} + V^i_a \Gamma^d_{ib} - V^i_b \Gamma^d_{ia}$, where W^d_{ab} is defined by (4.19). Thus, eq. (4.25) generates all the defining relations for the regular Lagrange structure $(\mathcal{E}, T, d_{\mathcal{E}})$ of type $(1, 1)$.

Resolving eqs. (4.20) step-by-step, we are obviously free to add to n-th order solution $\Omega^{(n)}$ any δ-closed (and hence exact) term. This ambiguity is not essential, however, as it can always be absorbed by a canonical transformation of the Poisson supermanifold N. The proof of the last fact is quite standard (see e.g. [2]), and we omit it here.

4.4 Exactness of Lagrange structure

An important observation about the weak antibracket (4.13), encoding the Lagrange structure, is that Ω_2 determines a trivial $\mathbb{D}$-cocycle. In other words, for any Ω_2 there is a function

$$\begin{gathered} G = G^{ij}(x)\bar{x}_i\bar{x}_j + \cdots\,, \\ \mathrm{Deg}(G) = 2\,, \qquad \epsilon(G) = 0\,, \qquad \mathrm{gh}(G) = 0\,, \end{gathered} \tag{4.26}$$

such that $\Omega_2 = \mathbb{D}G$. The last fact is a straightforward consequence of a more general statement about $\mathbb{D}$-cohomology.

Proposition 2. *Let $\mathcal{H}(\mathbb{D}) = \bigoplus \mathcal{H}^m_n(\mathbb{D})$ be the group of $\mathbb{D}$-cohomology (4.11), where the numbers m and n refer to the m-degree and the ghost number respectively, then $\mathcal{H}^m_n(\mathbb{D}) = 0$ for all $m > n$.*

Proof. We start with the following decomposition:

$$\mathbb{D} = \delta + \Delta\,, \tag{4.27}$$

where δ is given by (4.21) and

$$\deg(\delta) = -1\,, \qquad \deg(\Delta) \geq 0\,.$$

Thus, $\mathbb{D}$ is a deformation of δ by terms of higher r-degree and the statement follows immediately from the acyclicity of δ in positive r-degree. To make these arguments more explicit let us introduce the contracting homotopy for δ, i.e. an operator δ^* obeying the property

$$(\delta\delta^* + \delta^*\delta)A = A\,, \tag{4.28}$$

for all A with $\deg(A) > 0$. Using this operator, one can show that $\mathbb{D}$-cohomology is localized at zero r-degree. Indeed, applying δ^* to both sides of $\mathbb{D}A = 0$ yields

$$NA = \mathbb{D}\delta^* A\,, \qquad N \equiv 1 + (\delta^*\Delta + \Delta\delta^*)\,. \tag{4.29}$$

Since $\deg(\delta^*\Delta + \Delta\delta^*) > 0$, the operator N is invertible and commutes with $\mathbb{D}$. Thus $A = \mathbb{D}(N^{-1}\delta^* A)$. To complete the proof it remains to note that

$$\deg(A) \geq \mathrm{Deg}(A) - \mathrm{gh}(A)\,. \tag{4.30}$$

The last inequality can be verified just by comparing the lines of table 1. □

Applying now the inequality (4.30) to Ω_2, we get $\deg(\Omega_2) \geq 1$, and hence $\Omega_2 = \mathbb{D}G$. For a reason that will become clear later on, we call the function G, determined up to a $\mathbb{D}$-cocycle, a *propagator* associated to the weak anti-Poison structure Ω_2.

The proposition above allows us to give another prove of the existence theorem for the BRST charge Ω. Namely, consider the hamiltonian flow ϕ_t generated by the propagator G. Applying this flow to the symbol of homological vector field Ω_1, determined by rels. (4.8, 4.10), we get a one-parameter family of formal functions $\Omega(t) \in C^\infty(\mathcal{N})[[t]]$ related to each other by a formal canonical transform. By definition,

$$\Omega(t) = \phi_t^* \Omega_1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{G, \{G, \cdots, \{G, \Omega_1\} \cdots\} \,. \tag{4.31}$$

Clearly, the function $\Omega(1)$ obeys the boundary condition (4.17) and solves the master equation (4.7) whenever Ω_1 does. The problem thus reduces to solving the nilpotency condition for the homological vector field (4.10). Expanding $\Omega_1 = \sum \Omega_1^{(n)}$ according to the resolution degree, we get a chain of equations

$$\delta \Omega_1^{(n+1)} = K_n(\Omega_1^{(0)}, \ldots, \Omega_1^{(n)}) \,, \tag{4.32}$$

which are closely analogous to eqs. (4.20), though with δ which is different from (4.21). Here

$$\delta = T_a \frac{\partial}{\partial \eta_a} + \eta_a Z_A^a \frac{\partial}{\partial \xi_A} \tag{4.33}$$

is the usual Koszul-Tate differential associated with the shell Σ. The solvability of the system (4.32) can be easily seen from the acyclicity of δ in strictly positive r-degree [2] and the invariance of Σ under the gauge transformations (2.9).

4.5 Physical observables

Upon the BRST imbedding, the physical observables of original theory are usually identified with certain BRST cohomology in ghost number zero. Let us show that the space of physical observables $\mathcal{P}$, defined by (3.9), is naturally isomorphic to the subgroup $\mathcal{H}_0^0(\mathbb{D}) \subset \mathcal{H}(\mathbb{D})$ generated by the $\mathbb{D}$-cocycles of the ghost and momentum degree zero. Substituting the general expansion

$$F = \sum_{n=0}^{\infty} F^{(n)}(\varphi) = f(x) + \eta_a F_\alpha^a(x) c^\alpha + \xi_A F_{\alpha\beta}^A(x) c^\alpha c^\beta + \cdots \,, \quad \deg(F^{(n)}) = n \,, \tag{4.34}$$

to the $\mathbb{D}$-closedness condition $\mathbb{D}F = 0$ leads to the sequence of equations

$$\delta F^{(n+1)} = B_n(F^{(0)}, \ldots, F^{(n)}) \,, \qquad \deg(B_n) = n \,, \tag{4.35}$$

where the δ is given by (4.33). The first equation of this sequence reproduces the condition of on-shell invariance (3.7). Proceeding by induction in n and using the identity $\mathbb{D}^2 F = 0$, one can see that the r.h.s. of the n-th equation (4.35) is δ-closed, provided that eq. (3.7) is satisfied. Since the differential δ is acyclic in positive r-degree, we conclude that (i) any invariant function $f \in C(M)^{\mathrm{inv}}$ is lifted to a $\mathbb{D}$-cocycle $F \in C^\infty(\mathcal{N})$ with $\mathrm{gh}(F) = \mathrm{Deg}(F) = 0$, and (ii) any two equivalent (in the sense of (3.8)) functions $f_1, f_2 \in C^\infty(M)^{\mathrm{inv}}$ determine the same class of $\mathbb{D}$-cohomology upon the lift:

$$F_1 - F_2 = \mathbb{D}K \,, \qquad K = K^a \eta_a + \cdots \,. \tag{4.36}$$

This establishes an isomorphism between the space of physical observables $\mathcal{P}$ of the original theory (3.9) and the BRST cohomology group $\mathcal{H}_0^0(\mathbb{D})$.

4.6 Lagrange structure from the hamiltonian viewpoint.

In the conventional BFV approach the BRST charge arises as a tool for quantizing first-class constrained hamiltonian systems. Given a Poisson manifold P, the first class constraints $\Theta_I \in C^\infty(P)$ are defined as an (overcomplete) basis in the regular, Poisson-closed ideal of functions vanishing on a coisotropic submanifold $C \subset P$. In a more general setting [22], one can think of $\{\Theta_I\}$ as a section of a (nontrivial) vector bundle $E \to P$, which intersects the base P at points of C. The standard BFV-BRST theory [23, 2] corresponds to the case of a trivial vector bundle E. According to the general prescriptions of BFV-BRST theory, to each first class constraint, one has to associate a pair of canonically conjugated ghost variables $(\mathcal{C}, \mathcal{P})$, extending thus the original Poisson manifold P to the supermanifold $\Pi(E \oplus E^*)$. In the reducible case, i.e when the constraints Θ_I are functionally dependent, the additional pairs of canonically conjugated variables (ghosts-for-ghosts) must be introduced into the scheme [23, 2].

A glance at table 1 is enough to see that the spectrum of ghost numbers corresponds to hamiltonian first-class constrained system of a first-stage of reducibility. In order to make this interpretation explicit let us combine the local coordinates with ghost numbers 1 and -1 into the ghost coordinates $\mathcal{C}^I = (\bar{\eta}^a, c^\alpha)$ and ghost momenta $\mathcal{P}_I = (\eta_a, \bar{c}_\alpha)$, respectively. In this notation the BRST charge (4.17) can be rewritten as

$$\Omega = \mathcal{C}^I \Theta_I(x, \bar{x}) + \mathcal{P}_I \Xi^I_A(x, \bar{x}) \xi^A + \frac{1}{2} \mathcal{P}_K U^K_{IJ}(x, \bar{x}) \mathcal{C}^J \mathcal{C}^I + O(\mathcal{P}^2, \xi^2)\,, \tag{4.37}$$

where the expansion coefficients $\Theta_I = (\tilde{T}_a, \tilde{R}_\alpha)$ and $\Xi^I_A = (\tilde{Z}^a_A, 0)$, playing the role of first class constraints and their null-vectors, are given by the formal power series in $\bar{x}$'s of the form

$$\begin{aligned}
\tilde{T}_a(x, \bar{x}) &= T_a(x) + V^i_a(x)\bar{x}_i + O(\bar{x}^2)\,,\\
\tilde{R}_\alpha(x, \bar{x}) &= R^i_\alpha(x)\bar{x}_i + O(\bar{x}^2)\,,\\
\tilde{Z}^a_A(x, \bar{x}) &= Z^a_A(x) + O(\bar{x})\,.
\end{aligned} \tag{4.38}$$

At lowest orders in $\mathcal{C}$'s the master equation (4.7) gives the standard involution relations for a set of reducible first-class constraints

$$\{\Theta_I, \Theta_J\} = U^K_{IJ}\Theta_K\,, \qquad \Xi^I_A \Theta_I = 0\,, \tag{4.39}$$

w.r.t. the canonical Poisson bracket on T^*M. From the regularity condition it readily follows that the number of the independent first class constraints Θ_I is equal to $\dim M$. In physical terms, one can interpret this fact concluding that the considered hamiltonian system has no (local) physical degrees of freedom. From the geometrical viewpoint this implies that equations $\Theta_I = 0$ define a lagrangian submanifold $L \subset T^*M$; more accurately, L is a formal lagrangian submanifold as we are not concerned with convergence of the formal series (4.38).

The first class constraints Θ_I can also be regarded as a formal deformation of those given by the leading terms of expansions (4.38) in the "direction" of the Lagrange anchor V.

The involution relations for the "initial" constraints $T_a(x)$ and $R^i_\alpha(x)\bar{x}_i$ readily follow from nilpotency of the homological vector field (4.10). The integrability condition (2.7) results then from the requirement that the deformed constraints (4.38) have to be the first class as well. From this standpoint, the Lagrange structure can be understood as an infinitesimal of deformation of the lagrangian submanifold $L_0 \subset T^*M$ defined by the equations $T_a(x) = 0$ and $R^i_\alpha(x)\bar{x}_i = 0$. As is shown in the next section, any nonzero Lagrange anchor gives rise to quantum fluctuations of physical observables. In other words, any "classical" deformation of $L_0 \subset T^*M$ (in the category of lagrangian submanifolds in T^*M) results in a quantum deformation upon path-integral quantization.

Actually, the results of section 4.4 allows one to interpret the series (4.38) as Tailor's expansion in $\bar{x}$'s of some smooth functions $\Theta_I(x,\bar{x})$, $\Xi^I_A(x,\bar{x})$ defined in a sufficiently small vicinity U of any given point $p \in T^*M$. This can be proved as follows. Since both the symbol Ω_1 of homological vector field (4.10) and the generator G of canonical transform (4.31) are smooth function on $\mathcal{N}$ (with polynomial dependence of fiber variables), one can assert that for any $p \in \mathcal{N}$ there exists a neighborhood W together with $\varepsilon > 0$ such that $\Omega(t) = \phi_t\Omega_1$ is a smooth function on W for all $t \in [0,\varepsilon)$. Moreover, shrinking the vicinity $W \subset \mathcal{N}$ along the fibers, one can always choose $\varepsilon > 1$, so that

$$\Theta_I(x,\bar{x}) \equiv \left.\frac{\partial\Omega(1)}{\partial\mathcal{C}^I}\right|_{\mathcal{C}=\xi=0}, \qquad \Xi^I_A(x,\bar{x}) \equiv \left.\frac{\partial^2\Omega(1)}{\partial\xi^A\partial\mathcal{P}_I}\right|_{\mathcal{C}=\xi=0}$$

are smooth functions on an open subset $U = W \cap T^*M$.

Let $x_0 \in \Sigma \subset M$ be a classical solution and let U be a sufficiently small neighbourhood of x_0 in T^*M for which the equations $\Theta_I(x,\bar{x}) = 0$ make sense, i.e. determine a lagrangian submanifold $L \subset U$. If $\mathrm{rank}(V^i_a(x), R^i_\alpha(x)) = m$ for all $x \in U \cap M$, then one can split the position coordinates and momenta onto two groups, $x^i = (y^I, z^J)$ and $\bar{x}_i = (\bar{y}_I, \bar{z}_J)$, such that the index I runs m values, and the lagrangian submanifold $L \subset U$ is determined by the equations

$$\bar{y}_I = \frac{\partial\Psi(y,\bar{z})}{\partial y^I}, \qquad z^J = \frac{\partial\Psi(y,\bar{z})}{\partial\bar{z}_J}, \tag{4.40}$$

Ψ being the generating function for L of the first kind [26]. Indeed, due to the rank condition the equations $\Theta_I(x,\bar{x}) = 0$ can be explicitly resolved w.r.t. m of $\dim M$ momenta $\bar{x}_i$. Then using the rest of the equations one can express $\dim M - m$ position coordinates x's in terms of the other variables (as the total number of independent equations equals $\dim M$.) Since the resolved constraints are always in the abelian involution, one finds immediately that the r.h.s. of these constraints are to be given by the gradient of some function $\Psi(y,\bar{z})$.

Now we can use the local representation (4.40) for L to prove the announced proposition 1. Clearly, the classical equations of motion $T_a(x) = 0$ are equivalent to $\Theta_I(x,\bar{x}) = 0$ and $\bar{x}_i = 0$. (Geometrically speaking, the shell Σ is given by the intersection $L \cap M$.) Setting in eqs. (4.40), $\bar{x}_i = 0$ yields the following local representation for the shell:

$$\frac{\partial S(y)}{\partial y^I} = 0, \qquad z^J = E^J(y), \tag{4.41}$$

where

$$S(y) \equiv \Psi(y,0)\,, \qquad E^J(y) \equiv \frac{\partial \Psi}{\partial \bar{z}_J}(y,0)\,. \tag{4.42}$$

A simple linear algebra shows that the number $k = \dim M - m$ of non-lagrangian equations of motion is defined by the formula of proposition 1.

5. Quantization

The usual path-integral quantization deals with computing of quantum averages for the physical observables (= function(al)s on the space of trajectories M). The quantum average of an observable F is given by its integral over M with the uniform weight $e^{\frac{i}{\hbar}S}$, S being the action functional of the system. In the gauge invariant lagrangian theory, this simple rule is replaced by a more sophisticated BV scheme [1, 2] realizing the same idea in the presence of gauge invariance. If the original classical system admitted consistent operator BFV-BRST quantization, the BV method can be deduced [27] from the hamiltonian BFV-BRST scheme. In general, the relationship remains obscure between the BV quantization and the deformation quantization of the (weak) Poisson manifolds, so the BV method is presently viewed as an *ad hoc* postulate for the path-integral quantization. Obviously, the BV scheme can not be directly applied to quantize the systems having no action functional. In this section, we extend the path-integral quantization method to include not necessarily lagrangian (gauge) systems.

Let us briefly outline the quantization algorithm we suggest. The starting point is the BRST embedding for the Lagrange structure presented in the previous section. Upon this embedding, the physical observables of the original theory are identified with the BRST cohomology group $\mathcal{H}^0_0(\mathbb{D})$. As the next step, making use of the AKSZ method [13] in the form of ref. [14], we construct a topological sigma-model related to this BRST complex. Then we prove that the dynamics of this topological sigma-model are equivalent to the original classical theory; in so doing, the space of physical observables $\mathcal{H}^0_0(\mathbb{D})$ is naturally identified with the boundary observables of the topological sigma-model. The topological sigma-model, being a lagrangian theory in usual sense, can be quantized by the standard BV prescription, that results in quantizing the original theory which is not necessarily lagrangian. In a particular case of lagrangian systems, the sigma-model path-integral can be explicitly localized at the boundary, where it precisely reproduces the BV answer for the original lagrangian theory.

5.1 Topological sigma-model

Consider the $(1,1)$-dimensional supermanifold $\mathcal{I} = \Pi T I$ with boundary associated to the odd tangent bundle of the closed interval $I = [0,1] \subset \mathbb{R}$. The "points" of $\mathcal{I}$ are parameterized by one even coordinate $t \in [0,1]$ of ghost number 0 and one odd coordinate $\theta \in \Pi T_t I$ of ghost number 1. We shall also use the collective notation $z = (t,\theta)$. By the boundary of $\mathcal{I}$ we mean the two-point set $\partial\mathcal{I} = \{z_0, z_1\} \subset \mathcal{I}$ constituted by the "end points" $z_0 = (0,0)$ and $z_1 = (1,0)$ of the superinterval $\mathcal{I}$. The canonical volume element on $\mathcal{I}$ is given by $d^2z = dtd\theta$.

Consider now the superspace $\mathcal{N}^{\mathcal{I}}$ of all smooth maps from the source supermanifold $\mathcal{I}$ to the target supermanifold $\mathcal{N}$. (The latter is defined by (4.1).) In terms of local coordinates $\phi^k = (\varphi^I, \bar{\varphi}_J)$ on $\mathcal{N}$ each element $(\phi : \mathcal{I} \to \mathcal{N}) \in \mathcal{N}^{\mathcal{I}}$ defines (and is defined by) a field configuration

$$\phi^k(z) = \phi^k(t) + \theta \overset{*}{\phi}{}^k(t) . \tag{5.1}$$

According to the definitions above

$$\epsilon(\phi^k) = \epsilon(\overset{*}{\phi}{}^k) + 1 , \qquad \mathrm{gh}(\phi^k(t)) = \mathrm{gh}(\overset{*}{\phi}{}^k(t)) + 1 = \mathrm{gh}(\phi^k) . \tag{5.2}$$

The action of the topological sigma-model reads

$$\mathcal{S}[\phi] = \int_{\mathcal{I}} d^2z(\Lambda_k(\phi)\mathrm{D}\phi^k - \Omega(\phi)) . \tag{5.3}$$

The first and second terms are given here by the pull-backs of the symplectic potential (4.3) and the BRST charge (4.17) respectively, and

$$\mathrm{D} = \theta \frac{\partial}{\partial t} \tag{5.4}$$

is an odd, nilpotent vector field on $\mathcal{I}$ of ghost number 1. Taking into account the Grassman parity of all the factors entering the integrand (5.3) one can see that $\mathrm{gh}(\mathcal{S}) = 0$. The action (5.3) admits a straightforward BV interpretation that will be given in the next section 5.2.

The Poisson bracket (4.4) on $\mathcal{N}$ induces the antibracket (i.e. the odd Poisson bracket) on the superspace $\mathcal{N}^{\mathcal{I}}$. By definition,

$$(F, G) = \int_{\mathcal{I}} d^2z \left(\frac{\delta_r F}{\delta\phi^k(z)} \omega^{km}(\phi(z)) \frac{\delta_l G}{\delta\phi^m(z)} \right) , \tag{5.5}$$

for any functionals of fields $F[\phi]$ and $G[\phi]$.

The model (5.3) is called topological since the action $\mathcal{S}$ is required to satisfy the classical master equation

$$(\mathcal{S}, \mathcal{S}) = 0 . \tag{5.6}$$

An explicit calculation yields

$$\frac{1}{2}(\mathcal{S}, \mathcal{S}) = \int_{\mathcal{I}} d^2z \left(\mathrm{D}\Omega + \frac{1}{2}\{\Omega, \Omega\}(\phi(z)) \right) = \Omega(z_1) - \Omega(z_0) . \tag{5.7}$$

To meet the master equation, we impose the following boundary conditions on the momenta:

$$\bar{\varphi}^J|_{\partial\mathcal{I}} = 0 . \tag{5.8}$$

Then $\Omega(z_0) = \Omega(z_1) = 0$, as $\mathrm{Deg}(\Omega) \geq 1$ due to the definition (4.6).

As usual, the classical observables of the topological sigma-model (5.3) are identified with zero-ghost-number cohomology of the BRST differential $(\mathcal{S}, \cdot)$: The functional F defines a classical observable iff

$$\mathrm{gh}(F) = 0 \qquad \text{and} \qquad (\mathcal{S}, F) = 0 , \tag{5.9}$$

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and two BRST closed functionals F and G are considered to define the same classical observables ($F \sim G$) if they belong to the same class of BRST cohomology, i.e. $F - G = (\mathcal{S}, H)$ for some H.

Of particular interest are the *boundary observables*. These are constructed from the physical observables $[F] \in \mathcal{H}^0_0(\mathbb{D}) = \mathcal{P}$ of the original gauge theory (see section 4.5) by the rule $\hat{F}[\phi] = F(\varphi(z_1))$. A simple computation shows that

$$(\mathcal{S}, \hat{F}) = \{\Omega, F\}(\varphi(z_1)) = (\mathbb{D}F)(\varphi(z_1)) = 0\,. \tag{5.10}$$

If $F = \mathbb{D}G$, then $\hat{F} = (\mathcal{S}, \hat{G})$, and hence $\hat{F} \sim 0$. Clearly, replacing the point z_1 with z_0 one gets another set of classical observables supported on the other end of the superinterval $\mathcal{I}$.

The quantum average of a classical observable $[F] \in \mathcal{H}^0_0(\mathbb{D})$ corresponding to the original (not necessarily lagrangian) gauge theory $(\mathcal{E}, T, d_{\mathcal{E}})$ is defined by the path integral

$$\langle F \rangle = \int_{\mathcal{N}^{\mathcal{I}}} \mathcal{D}\phi F(\varphi(z_1)) e^{\frac{i}{\hbar}\mathcal{S}[\phi]}\,, \tag{5.11}$$

where the integration measure is normalized in such a way that $\langle 1 \rangle = 1$. To evaluate this path integral one has to impose a gauge fixing condition and choose an appropriate integration measure. As the sigma-model action entering (5.11) is a proper solution to the master equation, the gauge fixing procedure is standard for the BV method [1], i.e. this means to fix any Lagrange surface in the anti-Poisson manifold. The regularization of possible divergencies in (5.11) does not have any specificity compared to any other sigma-model path integral.

The main result of the section is Relation (5.11) defining quantum average for a physical observable of any (i.e. not necessarily lagrangian) dynamical system in terms of the usual path integral for the topological sigma-model. Below we elaborate on the equivalence between the original system and the topological sigma-model at the level of classical dynamics.

5.2 Classical equivalence

It was shown in refs. [14, 6] that the action of any topological sigma-model on $(1,1)$-dimensional supermanifold can be interpreted as the BV master action of a constrained hamiltonian system. In the case under consideration such a hamiltonian system has been already constructed, in fact, in section 4.6. To recover this effective hamiltonian constrained system from the action (5.3) one should integrate out of θ in (5.3) and set to zero all the fields with nonzero ghost number. The result will have the form

$$S_0[x, \bar{x}, \lambda] = \int_I (\bar{x}_i dx^i - \lambda^I \Theta_I(x, \bar{x}))\,. \tag{5.12}$$

where we introduced the notation $\lambda^I \equiv (\overset{*}{\bar{\eta}}{}^a, \overset{*}{c}{}^\alpha)$. Clearly, S_0 is nothing but the hamiltonian action functional on the cotangent bundle T^*M with the total hamiltonian given by the linear combination of the (reducible) first class constraints (4.38). Upon this identification the 1-forms λ^I on I play the role of Lagrange multipliers to Θ's. The action S_0 is invariant under the standard gauge transformation

$$\begin{aligned} \delta_\varepsilon x^i &= \{x^i, \Theta_I\}\varepsilon^I\,, \qquad \delta_\varepsilon \bar{x}_i = \{\bar{x}_i, \Theta_I\}\varepsilon^I\,, \\ \delta_\varepsilon \lambda^I &= d\varepsilon^I - \lambda^K U^I_{KJ}\varepsilon^J + \Xi^I_A \varepsilon^A\,, \end{aligned} \tag{5.13}$$

where $\varepsilon^I = (\varepsilon^a, \varepsilon^\alpha)$ and ε^A are the infinitesimal gauge parameters, and the structure functions U^I_{KJ}, Ξ^I_A are given by (4.39). The compatibility between the gauge transformations (6.11) and the boundary conditions $\bar{x}_i(0) = \bar{x}_i(1) = 0$ implies that $\varepsilon^I(0) = \varepsilon^I(1) = 0$. The linear dependence of constraints Θ_I leads to the linear dependence of the gauge algebra generators (5.13). Indeed, substituting

$$\varepsilon^I = \Xi^I_A \varrho^A\,, \qquad \varepsilon^A = d\varrho^A - \lambda^I W^A_{IB}\varrho^B \tag{5.14}$$

turns (5.13) to the trivial (on-shell vanishing) gauge transformation:

$$\begin{aligned} \delta_\varrho x^i &= \frac{\delta S_0}{\delta \lambda^I}\{x^i, \Xi^I_A\}\varrho^A\,, \qquad \delta_\varrho \bar{x}_i = \frac{\delta S_0}{\delta \lambda^I}\{\bar{x}_i, \Xi^I_A\}\varrho^A\,, \\ \delta_\varrho \lambda^I &= -\frac{\delta S_0}{\delta x^i}\{x^i, \Xi^I_A\}\varrho^A - \frac{\delta S_0}{\delta \bar{x}_i}\{\bar{x}_i, \Xi^I_A\}\varrho^A\,. \end{aligned} \tag{5.15}$$

Here we used the definition of the structure function W

$$\{\Xi^I_B, \Theta_J\} - \Theta^K_B U^I_{KJ} = W^A_{JB}\Xi^I_A \tag{5.16}$$

following from the identity $\{\Xi^I_A \Theta_I, \Theta_J\} = 0$.

Now the BV interpretation of the sigma-model action functional $\mathcal{S} = S_0 + \cdots$ becomes obvious: it is just the BV master action corresponding to the theory with reducible first class constraints and the action (5.12). Upon this interpretation, the fields $(\bar{\eta}^a, \overset{*}{\bar{\xi}}{}^A, c^\alpha)$ are identified with the ghosts corresponding to the infinitesimal gauge parameters $(\varepsilon^a, \varepsilon^A, \varepsilon^\alpha)$, $\bar{\xi}^A$ are ghosts for ghosts, and the other component fields are antifields to the aforementioned ones including the original gauge fields $(\overset{*}{\bar{\eta}}{}^a, \overset{*}{c}{}^\alpha, x^i, \bar{x}_i)$.

From the hamiltonian viewpoint, the model (5.12) has no (local) degrees of freedom as the first class constraints $\Theta_I = 0$ define a lagrangian submanifold $L \subset T^*M$. This amounts to saying that, given a "time"[6] moment $t_0 \in (0,1)$, by an appropriate gauge transformation (5.13) one can always move any point $(x^i(t_0), \bar{x}_j(t_0)) \in L$ to any other point of L assigning, simultaneously, any given value to $\lambda(t_0)$, no matter what were the boundary/initial values of these variables at $t = 0$ or $t = 1$. Whereas at the end points of the "time" interval we have the boundary conditions $\bar{x}_i(0) = \bar{x}_i(1) = 0$ reducing the constraints

$$0 = \frac{\delta S_0}{\delta \lambda^I} = \Theta_I(x, \bar{x})$$

(the only dynamical equations one has actually to solve) to the original equations of motion $T_a(x(0)) = T_a(x(1)) = 0$. So, we can conclude that the dynamical content of the topological sigma-model (5.3) is equivalent to that of the original (non-lagrangian) gauge theory.

[6] To avoid confusion, let us recall that it is the "time" t which is an auxiliary dimension introduced when the original dynamics (governed by the equations of motion (2.3)) is embedded into the topological sigma-model dynamics. The original equations of motion (2.3) (that are defined on the boundary, from the viewpoint of this sigma-model) contain their own evolution parameter.

6. Examples

6.1 BV field-antifield formalism

In this section we quantize the standard lagrangian gauge system by the proposed general method that works for not necessarily lagrangian theories. In this case, the standard BV quantization will be shown to follow from the proposed quantization scheme.

In the BV formalism the classical gauge theory is completely specified by master action $S(\phi)$ defined on an antisymplectic manifold $\mathcal{M}$ of fields and antifields ϕ^i and subject to the classical master equation

$$(S,S)_{\mathcal{M}} \equiv \frac{\partial_r S}{\partial\phi^I}\sigma^{IJ}\frac{\partial_l S}{\partial\phi^J} = 0\,, \tag{6.1}$$

σ^{IJ} being the odd bivector dual to the antisymplectic structure $\sigma = d\phi^I\sigma_{IJ}d\phi^J$. Besides the Grassman parity, the supermanifold $\mathcal{M}$ is graded by the ghost number and it is additionally required that

$$\mathrm{gh}(S) = 1\,, \qquad \mathrm{gh}((F,G)) = \mathrm{gh}(F) + \mathrm{gh}(G) - 1\,, \tag{6.2}$$

for any homogeneous $F, G \in C^\infty(\mathcal{M})$.

We start with the following simple observation which might be of some interest in its own right: *Any lagrangian gauge theory on $\mathcal{M}$ with the master action $S(\phi)$ admits an equivalent reformulation as the topological sigma-model having $\mathcal{M}$ as target manifold.* The construction is as follows. Let $\mathcal{J} = \mathcal{I}\times\Pi\mathbb{R}$ be the $(1,2)$-dimensional supermanifold with boundary, given by the direct product of the superinterval $\mathcal{I} = \Pi TI$ with coordinates $z = (t,\theta)$ and the odd linear space $\Pi\mathbb{R}$ "parameterized" by $\bar\theta$. The ghost number assignments are given by

$$\mathrm{gh}(t) = 0\,, \qquad \mathrm{gh}(\theta) = 1\,, \qquad \mathrm{gh}(\bar\theta) = -1\,. \tag{6.3}$$

The "points" of $\mathcal{J}$ are thus the triples $u = (t,\theta,\bar\theta)$, where $t\in[0,1]$. The boundary of $\mathcal{J}$ is, by definition, a two point set $\partial\mathcal{J} = \{u_0, u_1\}$ constituted by $u_0 = (0,0,0)$ and $u_1 = (1,0,0)$.

The field content of the topological sigma-model in question is identified with the superspace $\mathcal{M}^{\mathcal{J}}$ of maps from $\mathcal{J}$ to $\mathcal{M}$. In terms of local coordinates ϕ^I on $\mathcal{M}$, each $\phi\in\mathcal{M}^{\mathcal{J}}$ is described by a field configuration

$$\phi^I(u) = \varphi^I(z) + \bar\theta\bar\varphi^I(z) = \varphi^I(t) + \theta\overset{*}{\varphi}{}^I(t) + \bar\theta\bar\varphi^I(t) + \bar\theta\theta\overset{*}{\bar\varphi}{}^I(t)\,, \tag{6.4}$$

Observe that under the general coordinate transformations on $\mathcal{M}$, $\phi^I \to \tilde\phi^I(\phi)$, the component superfields $\{\bar\varphi^I(z)\}$ behave like the coordinates of a tangent vector to $\mathcal{M}$. So, one may think of the superfields $\phi^I(u) = (\varphi^I(z), \bar\varphi^I(z))$ as smooth maps from the total space of the odd tangent bundle ΠTI to that of $\Pi T\mathcal{M}$.

Let us suppose for a moment that the antisymplectic 2-form is exact i.e. $\sigma = d\rho$ as it happens in the conventional BV theory. Then we can define the following action of the topological sigma-model

$$\mathcal{S} = \int_{\mathcal{J}} d^3u(\rho_I(\phi)\bar{\mathrm{D}}\phi^I - S(\phi))\,. \tag{6.5}$$

Here $d^3u = dtd\theta d\bar{\theta}$ is a natural integration measure on $\mathcal{J}$, and

$$\bar{\mathrm{D}} = \frac{\partial}{\partial\bar{\theta}} + \mathrm{D}\,, \qquad \mathrm{D} = \theta\frac{\partial}{\partial t} \tag{6.6}$$

are odd, self-commuting vector fields on $\mathcal{J}$ of ghost number 1. Notice that $\mathrm{gh}(\mathcal{S}) = 0$. As will be shown below, action (6.5) is equivalent, on the one hand, to the general topological sigma-model action (5.3) constructed for the original lagrangian equations of motion with the anchor being the unit matrix, and on the other hand to the BV master action for the original lagrangian theory.

The antibracket on $\mathcal{M}$ gives rise to that on $\mathcal{M}^{\mathcal{J}}$:

$$(\phi^I(u), \phi^J(u'))_{\mathcal{M}^{\mathcal{J}}} = \sigma^{IJ}(\phi(u))\delta^3(u-u')\,. \tag{6.7}$$

Taking the antibracket of the action (6.5) with itself, we find

$$(\mathcal{S},\mathcal{S})_{\mathcal{M}^{\mathcal{J}}} = \int d^3u\left(\frac{\delta_r\mathcal{S}}{\delta\phi^I(u)}\sigma^{IJ}(\phi(u))\frac{\delta_l\mathcal{S}}{\delta\phi^J(u)}\right) = \frac{\partial\phi^I}{\partial\bar{\theta}}\left(\sigma_{IJ}\frac{\partial\phi^J}{\partial\bar{\theta}} - \partial_I S\right)\Bigg|_{u_0}^{u_1}. \tag{6.8}$$

To cancel the boundary terms in the r.h.s. of the last expression we can set

$$\frac{\partial\phi^I}{\partial\bar{\theta}}\bigg|_{\partial\mathcal{J}} = 0 \quad \Leftrightarrow \quad \bar{\varphi}^I(0) = \bar{\varphi}^I(1) = 0\,. \tag{6.9}$$

With these boundary conditions the functional $\mathcal{S}$ becomes the classical master action on the antisymplectic supermanifold $\mathcal{M}^{\mathcal{J}}$. Integrating out of $\bar{\theta}$ we get

$$\mathcal{S} = \int_{\mathcal{I}} d^2z\left(\bar{\varphi}^I\sigma_{IJ}\mathrm{D}\varphi^I + \frac{1}{2}\bar{\varphi}^I\sigma_{IJ}\bar{\varphi}^J + \bar{\varphi}^I\frac{\partial S}{\partial\varphi^I}\right). \tag{6.10}$$

As is seen, the functional $\mathcal{S}$ depends actually on the antisymplectic structure σ, but not on the choice of antisymplectic potential Λ.

In consequence of the master equation the action (6.10) is invariant under the abelian gauge transformations

$$\begin{aligned}
\delta_\varepsilon\varphi^I &= \varepsilon^I\,,\\
\delta_\varepsilon\bar{\varphi}^J &= -\mathrm{D}\varepsilon^J + \varepsilon^L\frac{\partial^2 S}{\partial\varphi^L\partial\varphi^I}\sigma^{IJ} + \varepsilon^L\frac{\partial\sigma_{IK}}{\partial\varphi^L}(\bar{\varphi}^K + \mathrm{D}\varphi^K)\sigma^{IJ}\,,
\end{aligned} \tag{6.11}$$

with the infinitesimal gauge parameter $\varepsilon^I(z)$ subject to the boundary conditions

$$\varepsilon^I(z)\partial_I S(\varphi(z))|_{\partial\mathcal{I}} = 0\,. \tag{6.12}$$

Since S is a proper solution to the classical master equation (6.1), only "half" of the gauge parameters $\varepsilon^i(z)$ need vanish at the boundary points z_0 and z_1.

Notice that the action (6.10) is identical in form to the action (5.3). Making identifications

$$\Lambda = \bar{\varphi}^I\sigma_{IJ}d\varphi^J\,, \qquad \Omega = \Omega_1 + \Omega_2 = \bar{\varphi}^I\frac{\partial S}{\partial\varphi^I} + \frac{1}{2}\bar{\varphi}^I\sigma_{IJ}\bar{\varphi}^J\,, \tag{6.13}$$

one can readily check that the BRST charge Ω obeys the master equation $\{\Omega,\Omega\}=0$ with the Poisson bracket corresponding to the exact symplectic structure $\omega = d\Lambda$. Moreover, when S is the minimal sector master action of an irreducible gauge theory, the spectrum of component fields $(\varphi^I,\bar{\varphi}^J)$ coincides with that presented in table 1. We leave the check of details to the reader.

Notice that the "truncated" BRST charge $\Omega_1 = \bar{\varphi}^I\partial S/\partial\varphi^I$ was first considered in ref. [14]. In that paper, a very convenient superfield technique was developed to illuminate the relationship between the BFV-BRST charge and the BV master action. In this section we use a similar technique, although the same results can be derived without superfields.

According to the results of section 4.2, the term Ω_1 defines a homological vector field on $\mathcal{M}$:

$$\begin{aligned} QF = \{\Omega_1, F\} &= \frac{\partial_r S}{\partial\varphi^I}\sigma^{IJ}\frac{\partial_l F}{\partial\varphi^J}\,, \qquad \forall F\in C^\infty(\mathcal{M})\,,\\ Q^2 = 0 \quad &\Leftrightarrow \quad \{\Omega_1,\Omega_1\} = 0\,. \end{aligned} \tag{6.14}$$

In the case at hand the operator Q is nothing but the usual BV-differential associated with the classical master action S. The second term Ω_2 defines (and is defined by) the original anti-Poisson structure on $\mathcal{M}$:

$$\begin{aligned} (F,G) = \{\{\Omega_2,F\},G\}\,, &\qquad \forall F,G,H\in C^\infty(\mathcal{M})\,,\\ (-1)^{\epsilon(F)\epsilon(H)}(F,(G,H)) + cycle(F,G,H) = 0 \quad &\Leftrightarrow \quad \{\Omega_2,\Omega_2\} = 0\,. \end{aligned} \tag{6.15}$$

Both structures are compatible in the sense of the graded Liebnitz rule

$$Q(F,G) = (QF,G) + (-1)^{\epsilon(F)+1}(F,QG) \quad \Leftrightarrow \quad \{\Omega_1,\Omega_2\} = 0\,. \tag{6.16}$$

As we have shown in section 4.4, the last relation implies the existence of a function $G = G^{IJ}\bar{\varphi}_I\bar{\varphi}_J$, called the propagator, such that

$$\Omega_2 = \{\Omega_1, G\}\,, \quad \epsilon(G) = 0\,, \quad \mathrm{gh}(G) = 0\,. \tag{6.17}$$

Here we set $\bar{\varphi}_I \equiv \sigma_{IJ}\bar{\varphi}^J$, so that $\epsilon(\bar{\varphi}_I) = \epsilon(\varphi^I)$ and $\mathrm{gh}(\bar{\varphi}_I) = -\mathrm{gh}(\varphi^I)$. When $S(x)$ is the action of a system without gauge symmetry, the only nonzero block of the matrix G^{IJ} is given by the inverse to the van Vleck matrix $\partial_i\partial_j S(x)$ that justifies the term "propagator".

To establish a classical correspondence between the topological sigma-model (6.5) and the original gauge theory we first observe that according to (6.11) the fields φ^I are purely gauge ones in the interior of $\mathcal{I}$, while the fields $\bar{\varphi}^I$ enter to the action functional $\mathcal{S}$ only in an algebraic way (i.e. without derivatives). This suggests that all the dynamical degrees of freedom are supported at the boundary of the superinterval $\mathcal{I}$. Notice also that the form of the action (6.10) is quite similar to that of hamiltonian action functional with $\bar{\varphi}^I$ playing the role of momenta conjugated to the coordinates φ^I. So, to obtain the action functional governing the dynamics of the boundary degrees of freedom we can just eliminate the auxiliary fields $\bar{\varphi}^I$ from $\mathcal{S}$ by means of their own equations of motion:

$$\frac{\delta\mathcal{S}}{\delta\bar{\varphi}^I} = 0 \quad \Leftrightarrow \quad \bar{\varphi}^I = \mathrm{D}\varphi^I + \sigma^{IJ}\frac{\partial S}{\partial\varphi^J}\,. \tag{6.18}$$

As would be expected, the boundary field configurations $\varphi^I(z_{0,1})$ define the stationary points of S, since $\bar{\varphi}|_{\partial\mathcal{I}} = \mathrm{D}\varphi|_{\partial\mathcal{I}} = 0$. Substituting (6.18) to (6.10), we finally get

$$\mathcal{S}|_{\delta\mathcal{S}/\delta\bar{\varphi}=0} = \int_{\mathcal{I}} d^2z \mathrm{D}S = S(\varphi(z_1)) - S(\varphi(z_0)) \,. \tag{6.19}$$

The action describes two uncoupled copies of the original gauge theory (one for each end of the superinterval $\mathcal{I}$) with $\mathcal{M} \times \mathcal{M}$ being the total configuration space.

Let us now comment on quantum equivalence. Proceeding to quantization, one assigns the BV configuration space $\mathcal{M}$ with a nondegenerate density ρ and replaces the classical master equation (6.1) with the quantum one:

$$(S, S)_{\mathcal{M}} = 2i\hbar\Delta_{\mathcal{M}} S \,. \tag{6.20}$$

Here $\Delta_{\mathcal{M}} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is the odd Laplace operator defined by the rule

$$\Delta_{\mathcal{M}} F = \mathrm{div}_\rho X_F \,, \tag{6.21}$$

$X_F = (F, \ \cdot\ )_{\mathcal{M}}$ being the hamiltonian vector field corresponding to $F \in C^\infty(\mathcal{M})$. The density ρ is chosen in such a way that $\Delta^2_{\mathcal{M}} = 0$. By definition, a quantum observable is a function of fields $F(\phi)$ annihilated by the quantum BRST operator $\hat{S}_\hbar$:

$$\hat{S}_\hbar F = (S, F)_{\mathcal{M}} - i\hbar\Delta_{\mathcal{M}} F = 0 \,. \tag{6.22}$$

The quantum average of F is defined by the path integral

$$\langle F \rangle_S = \int_{\mathcal{M}} \mathcal{D}\phi\, \delta(\gamma_a(\phi)) F(\phi) e^{\frac{i}{\hbar}S(\phi)} \,. \tag{6.23}$$

where $\mathcal{D}\phi$ is the integration measure associated to ρ and equations

$$\gamma_a(\phi) = 0 \tag{6.24}$$

define an appropriate Lagrange surface in $\mathcal{M}$. (In the conventional BV scheme the constraints (6.24) are required to be in abelian involution, i.e. $(\gamma_a, \gamma_b)_{\mathcal{M}} = 0$, though, upon some modifications [28], a more general involution is also allowed.)

As with the antibracket, the measure density ρ on $\mathcal{M}$ induces that on the space of fields $\mathcal{M}^I$:

$$\tilde{\rho} = \prod_{u \in \mathcal{J}} \rho(\phi(u)) \,. \tag{6.25}$$

Then the functional counterpart of the odd Laplacian (6.21) reads

$$\Delta_{\mathcal{M}^{\mathcal{I}}} = \int_{\mathcal{J}} d^3u \tilde{\rho}^{-1} \frac{\delta}{\delta\phi^I(u)} \tilde{\rho} \sigma^{IJ}(\phi(u)) \frac{\delta}{\delta\phi^J(u)} \,. \tag{6.26}$$

Given the quantum master action S, we define the action of the topological sigma-model by the same formula (6.5). The latter is proved to be a solution to the quantum master equation on $\mathcal{M}^{\mathcal{I}}$ with renormalized Plank constant $\hbar'$. Indeed, a straightforward computation yields

$$(\mathcal{S},\mathcal{S})_{\mathcal{M}^{\mathcal{I}}} - 2i\hbar'\Delta_{\mathcal{M}^{\mathcal{I}}}\mathcal{S} = \int_{\mathcal{M}} d^3u((S,S)_{\mathcal{M}} - 2iC\hbar'\Delta_{\mathcal{M}})(\phi(u))\,, \tag{6.27}$$

where $C = \delta^3(0)$ is indefinite "constant". The functional $\mathcal{S}$ will satisfy the quantum master equation if we set $\hbar' = \hbar C^{-1}$. Notice that formally $\epsilon(C) = 0$ and $\mathrm{gh}(C) = 0$. To assign a precise meaning for the value $\delta^3(0)$ one has to apply a suitable regularization to the ill-defined Laplace operator (6.26). Similarly, after the renormalization above any quantum observable $F(\phi)$ of the original gauge theory gives rise to the boundary quantum observable $\hat{F}[\phi] = F(\phi(u_1))$ of the topological sigma-model, i.e. $\hat{\mathcal{S}}_{\hbar'}F(\phi(u_1)) = 0$.

To calculate the quantum average of a boundary observable $\hat{F}$ we can apply the Faddeev-Popov recipe to the naive path integral (6.35). This includes several steps. First one promotes the infinitesimal gauge parameter $\varepsilon^I(z)$ to a ghost field $\mathcal{C}^I(z)$ with opposite Grassman parity. Then one looks for an appropriate gauge fixing conditions. The explicit structure of the gauge transformations (6.11) suggests to impose conditions only on the fields $\varphi^I(z)$. This can be done in many (equivalent) ways. For example, choosing a symmetric connection ∇ on $\mathcal{M}$ we can set

$$\chi^I(z) \equiv \frac{\partial\varphi^I}{\partial\theta} + \theta(\ddot{\varphi}^I - \Gamma^I_{JK}(\varphi)\dot{\varphi}^J\dot{\varphi}^K) = 0\,. \tag{6.28}$$

Here Γ^I_{JK} are the Cristoffel symbols of ∇ and the overdot stands for the derivative in t. Let us assume that any two points of affine manifold $(\mathcal{M},\nabla)$ are connected by a unique geodesics. Then equations $\chi^I(z) = 0$ fix the superfield $\varphi^I(z)$ up to arbitrary boundary values $\varphi^I(z_0), \varphi^I(z_1) \in \mathcal{M}$. To fix the residual gauge symmetry at the boundary we may use the conditions (6.24):

$$\gamma_a(\varphi(z_0)) = \gamma_a(\varphi(z_1)) = 0\,. \tag{6.29}$$

Finally, to provide the correct integration measure in the path integral (5.11) one introduce the antighost fields $\bar{\mathcal{C}}_J$. By definition, $\epsilon(\bar{\mathcal{C}}_I) = \epsilon(\mathcal{C}^I)$. Geometrically, the fields $\mathcal{C}^I$ are $\bar{\mathcal{C}}_J$ can be viewed as taking values in tangent and cotangent spaces (with reverse parity) of the target manifold of fields φ's.

Since we regard the ghost-antighosts fields $\mathcal{C}$'s and $\bar{\mathcal{C}}$'s to be related with gauge symmetry in the interior of the superinterval $\mathcal{I}$, the appropriate boundary conditions for them are

$$\mathcal{C}^I|_{\partial\mathcal{I}} = 0\,, \qquad \bar{\mathcal{C}}_J|_{\partial\mathcal{I}} = 0\,. \tag{6.30}$$

Notice that we do not introduce ghost and antighost fields associated to the residual gauge symmetry at the boundary as such fields are assumed to be already included into the action S and the gauges γ's.

After all these preparations we can write

$$\langle F\rangle_{\mathcal{S}} = \int_{\mathcal{M}^{\mathcal{I}}} \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}\mathcal{C}\mathcal{D}\bar{\mathcal{C}}\varrho[\varphi]\delta[\chi(\varphi)]\delta[\gamma(\varphi(z_0))]\delta[\gamma(\varphi(z_1))]F(\varphi(z_1))e^{\frac{i}{\hbar}\mathcal{S}_{FP}}\,. \tag{6.31}$$

Here

$$\mathcal{S}_{FP}[\varphi,\bar{\varphi},\mathcal{C},\bar{\mathcal{C}}]=\mathcal{S}[\varphi,\bar{\varphi}]+\mathcal{S}_{\rm gh}[\varphi,\mathcal{C},\bar{\mathcal{C}}] \tag{6.32}$$

is the usual Faddeev-Popov action given by the sum of the initial gauge invariant action and the ghost action

$$\begin{aligned}\mathcal{S}_{\rm gh}\equiv\int_{\mathcal{I}}d^2z\,\bar{\mathcal{C}}_I(\delta_{\mathcal{C}}\chi^I)|_{\chi=0}=\int_{\mathcal{I}}d^2z\left(\bar{\mathcal{C}}^I\frac{\partial\mathcal{C}_I}{\partial\theta}+\theta(\nabla_t\mathcal{C}^I\nabla_t\bar{\mathcal{C}}_I+R^L_{IJK}(\varphi)\mathcal{C}^I\dot{\varphi}^J\dot{\varphi}^K\bar{\mathcal{C}}_L)\right),\\ \nabla_t\mathcal{C}^I=\dot{\mathcal{C}}^I-\dot{\varphi}^J\Gamma^I_{JK}(\varphi)\mathcal{C}^K\,,\qquad\nabla_t\bar{\mathcal{C}}_I=\dot{\bar{\mathcal{C}}}_I-\dot{\varphi}^J\Gamma^K_{JI}(\varphi)\bar{\mathcal{C}}_K\,.\end{aligned} \tag{6.33}$$

If one disregards the Grassman nature of the Faddeev-Popov ghosts $\mathcal{C}$'s and $\bar{\mathcal{C}}$'s, then the second term in (6.33) is nothing but the Jacobi action for the deviation of geodesics. As to the factor

$$\varrho[\varphi]\equiv\mathrm{sdet}(\sigma^{IJ}(\varphi(z))\delta^2(z-w)), \tag{6.34}$$

it is introduced to provide the invariance of the integration measure under the gauge transformations (6.11). The appearance of this factor can be rigourously justified within BV scheme, but we shall not dwell on this here.

Notice that the path integral is Gaussian in the variables $\bar{\varphi}$, $\mathcal{C}$ and $\bar{\mathcal{C}}$, and assumes no actual integration over the interior values of φ's due to the gauge fixing conditions. Integrating successively over all these variables we get

$$\begin{aligned}\langle F\rangle_{\mathcal{S}}&=\int_{\mathcal{M}\times\mathcal{M}}\mathcal{D}\varphi(z_0)\mathcal{D}\varphi(z_1)\delta[\gamma(\varphi(z_0))]\delta[\gamma(\varphi(z_1))]F(\varphi(1))e^{\frac{i}{\hbar}(S(\varphi(z_1))-S(\varphi(z_0)))}\\ &=(\mathrm{const})\int_{\mathcal{M}}\mathcal{D}\varphi(z_1)\delta[\gamma(\varphi(z_1))]F(\varphi(z_1))e^{\frac{i}{\hbar}S(\varphi(z_1))}\,.\end{aligned} \tag{6.35}$$

(The role of the Faddeev-Popov ghosts was to compensate the Berezinian resulting from integration of the delta-functional $\delta[\chi(\varphi)]$ and the factor (6.34) was exactly compensated by the integration of $\bar{\varphi}$'s.) Including the inessential overall constant in (6.35) to the integration measure, we arrive at the desired equality

$$\langle F\rangle_S=\langle\hat{F}\rangle_{\mathcal{S}}\,, \tag{6.36}$$

where the $\hat{F}=F(\phi(u_1))$ is the boundary observable of the topological sigma-model (6.5) constructed from the quantum observable $F(\phi)$ of the original gauge theory (6.1). The net result, seen from (6.36), is that the path integral quantization based on the embedding into the topological sigma-model (5.3) (that does not require the original equations of motion to be lagrangian) in the lagrangian case brings precisely the same average values for the observables as in the standard BV-quantization.

6.2 General Lagrange structure of type (0,0)

In the previous section we have exemplified the quantization method applying it to a gauge system whose equations of motion are lagrangian. In this section, we apply the method to a complementary, in a sense, particular case: the system without gauge symmetry and with independent but general (i.e. not necessarily lagrangian) equations of motion. As will be seen, the quantization method works well in this case too, bringing the reasonable results admitting clear physical interpretation.

Let $(\mathcal{E}, T, d_{\mathcal{E}})$ be a Lagrange structure associated to a set of independent equations of motion $T_a(x) = 0$, so that the matrix $\partial_i T_a$ is on-shell nondegenerate. According to the general prescription of section 4, this classical theory is BRST embedded to the Poisson supermanifold. For the sake of simplicity we assume here that the dynamics bundle $\mathcal{E}$ admits a flat connection $\nabla = \partial$. The action of the topological sigma-model has the following structure:

$$\mathcal{S} = \int d^2 z \left(\bar{x}_i D x^i + \bar{\eta}^a D\eta_a - \Omega \right) = S_0 + (\text{ghost terms}) \,, \tag{6.37}$$

where

$$S_0 = \int_0^1 (\bar{x}^i dx^i - \lambda^a \tilde{T}_a(x, \bar{x})), \qquad \lambda^a \equiv \overset{*}{\bar{\eta}}{}^a \,, \tag{6.38}$$

is a hamiltonian action associated to the first class constraints (cf. (4.38))

$$\tilde{T}(x, \bar{x}) = T_a(x) + V_a^i(x)\bar{x}_i + O(\bar{x}^2) \,, \qquad \{\tilde{T}_a, \tilde{T}_b\} = U_{ab}^c \tilde{T}_c \,. \tag{6.39}$$

Let us further assume that the classical master action (6.37) meets also the quantum master equation (6.20) or, what is the same, that $\mathcal{S}$ is annihilated by the (suitably regularized) odd Laplace operator.

In order to write the gauge fixed action we then introduce the non-minimal sector of BV fields: the antighosts ζ_a and the Lagrange multipliers π_a as well as the corresponding antifields $\overset{*}{\zeta}{}^a$ and $\overset{*}{\pi}{}^a$. The Grassman parity and the ghost number assignments of the introduced variables are

$$\begin{aligned} &\epsilon(\zeta_a) = 1 \,, && \epsilon(\overset{*}{\zeta}{}^a) = 0 \,, && \epsilon(\pi_a) = 0 \,, && \epsilon(\overset{*}{\pi}{}^a) = 1 \,, \\ &\mathrm{gh}(\zeta_a) = -1 \,, && \mathrm{gh}(\overset{*}{\zeta}{}^a) = 0 \,, && \mathrm{gh}(\pi_a) = 0 \,, && \mathrm{gh}(\overset{*}{\pi}{}^a) = -1 \,. \end{aligned} \tag{6.40}$$

The explicit form of the gauge transformation (5.13) suggests to impose the following gauge-fixing condition on λ's:

$$\frac{d}{dt}(\lambda^a(e)) = 0 \,, \tag{6.41}$$

with e being a nowhere vanishing vector field on $[0, 1]$. The gauge fixing fermion associated to (6.41) is given by

$$\Psi = \int_0^1 dt \zeta_a \dot{\lambda}^a \,, \qquad \mathrm{gh}(\Psi) = -1 \,, \tag{6.42}$$

where we set for simplicity $e = \partial_t$. The standard non-minimal BV action $\mathcal{S} + \int_0^1 dt \pi_a \overset{*}{\zeta}{}^a$ depends thus on the fields

$$\phi^A = (x^i, \bar{x}_i, \lambda^a, \pi_a, \eta_a, \bar{\eta}^a, \zeta_a)$$

and antifields

$$\phi^*_A = (\overset{*}{x}{}^i, \overset{*}{\bar{x}}{}_i, \overset{*}{\lambda}{}^a, \overset{*}{\pi}{}_a, \overset{*}{\eta}_a \overset{*}{\bar{\eta}}{}^a, \overset{*}{\zeta}_a) \,.$$

Now the gauge fixed action is obtained by restricting the non-minimal BV action to the lagrangian submanifold $\mathcal{L}$:

$$\phi^*_A = \frac{\partial \Psi}{\partial \phi^A}\,. \tag{6.43}$$

The last equations allow one to express all the antifields via the fields. The result is

$$\mathcal{S}_{\rm gf} = \int_0^1 dt\left(\bar{x}_i\dot{x}^i + \pi_a\dot{\lambda}^a + \dot{\bar{\eta}}^a\zeta_a + \lambda^a\tilde{T}_a(x,\bar{x}) - \lambda^a\bar{\eta}^b U^c_{ab}(x,\bar{x})\dot{\zeta}_c\right). \tag{6.44}$$

The quantum average of the boundary observable $F(x(1))$ is defined now by the regularized version of the naive path integral (5.11)

$$\langle F\rangle = \int_{\mathcal{L}} F e^{\frac{i}{\hbar}\mathcal{S}_{\rm gh}}\,. \tag{6.45}$$

To elucidate the meaning of the last formula it is instructive to consider the case of trivial Lagrange structure:

$$V^i_a = 0\,, \qquad \tilde{T}_a(x,\bar{x}) = T_a(x)\,, \qquad U^c_{ab} = 0\,. \tag{6.46}$$

Integrating in (6.45) of $\bar{x}^i$, π_a, $\bar{\eta}^a$ and ζ_a one finds immediately

$$\langle F\rangle \sim \int \mathcal{D}x\mathcal{D}\lambda\delta[\dot{x}^i]\delta[\dot{\lambda}^a]F(x(1))e^{\frac{i}{\hbar}\int\lambda^a T_a(x)} \sim \int_M d^n x\delta(T_a(x))F(x) \sim F(x_0) \tag{6.47}$$

where x_0 is a unique solution to the classical equations of motion $T_a(x) = 0$. Normalizing the integration measure in such a way that $\langle 1\rangle = 1$, we can finally write $\langle F\rangle = F(x_0)$. We see that the quantum average of F involves no quantum corrections in $\hbar$ and coincides with value of functional $F(x)$ on a given classical trajectory $x_0 \in M$. Thus one can regards rel. (6.47) as a classical vacuum-vacuum amplitude in the presence of observable. In the context of hamiltonian mechanics such amplitudes were introduced and studied in ref. [19].

In order to relate the above result with conventional formulas of quantum mechanics let us consider the intermediate possibility: the anchor is degenerate but regular at the vicinity of a classical solution x_0. In this case, due to proposition 1 we can assume the equations of motion to have the form

$$\partial_I S(y) = 0\,, \qquad z^J = E^J(y)\,. \tag{6.48}$$

For these equations we have the canonical Lagrange anchor $V = (V^J, V_I)$, where the vector fields

$$V^J = 0\,, \qquad V_I = \frac{\partial}{\partial y^I} + \frac{\partial E^J}{\partial y^I}\frac{\partial}{\partial z^J} \tag{6.49}$$

form an abelian distribution. The integrability conditions (2.7) are obviously satisfied with C's equals zero. Substituting these data to the gauge fixed action (6.44), we arrive at Gaussian path integral for the quantum average (6.45). As in the previous case the path integral is localized at the boundary. The calculation is rather simple, so we just write the final result

$$\langle F\rangle \sim \int dy F(y,E(y))e^{\frac{i}{\hbar}S(y)} = \int dydz F(y,z)\delta\left(z^J - E^J(y)\right)e^{\frac{i}{\hbar}S(y)}\,. \tag{6.50}$$

Here $y^I = y^I(1)$, $z^J = z^J(1)$. The quantum average is given thus by a superposition of the classical amplitude (6.47) for the "non-lagrangian" degrees of freedom z's and the usual Feynman's amplitude associated with the partial action $S(y)$.

In the most general case of irregular Lagrange anchor we can use the Feynman perturbation technique to obtain the quasi-classical expansion for the quantum average around the classical solution

$$x_0 \in \Sigma\,, \qquad \bar{x} = 0\,, \qquad \lambda = 0\,, \qquad \pi = 0\,, \qquad \bar{\eta} = 0\,, \qquad \zeta = 0\,.$$

Thus we write $x(t) = x_0 + y(t)$, with a fluctuation field $y(t)$, and decompose the gauge fixed action on a free part and interaction, $\mathcal{S}_{\mathrm{gf}} = \mathcal{S}_0 + \mathcal{S}_{\mathrm{int}}$, with

$$\begin{aligned}
\mathcal{S}_0 &= \int_0^1 dt \left(\bar{x}_i \dot{y}^i + \pi_a \dot{\lambda}^a + \dot{\bar{\eta}}^a \dot{\zeta}_a + \lambda^a \partial_i T_a(x_0) y^i \right), \\
\mathcal{S}_{\mathrm{int}} &= \int_0^1 dt \Bigg(\lambda^a V_a^i(x_0)\bar{x}_i + \lambda^a \sum_{k+l>1} \frac{1}{k!l!} \partial_{i_1}\cdots\partial_{i_k}\partial^{j_1}\cdots\partial^{j_l}\tilde{T}_a(x_0,0) y^{i_1}\cdots y^{i_k}\bar{x}_{j_1}\cdots\bar{x}_{j_l} - \\
&\qquad - \lambda^a \bar{\eta}^b \dot{\zeta}_c \sum_{k,l=0}^{\infty} \frac{1}{k!l!} \partial_{i_1}\cdots\partial_{i_k}\partial^{j_1}\cdots\partial^{j_l} U_{ab}^c(x_0,0) y^{i_1}\cdots y^{i_k}\bar{x}_{j_1}\cdots\bar{x}_{j_l} \Bigg). \qquad (6.51)
\end{aligned}$$

The Feynman propagator is then deduced from the $\mathcal{S}_0$. With account of the boundary conditions

$$\begin{aligned}
\bar{x}^i(0) &= \bar{x}^i(1) = 0\,, \qquad \bar{\eta}^a(0) = \bar{\eta}^a(1) = 0\,, \\
\pi_a(0) &= \pi_a(1) = 0\,, \qquad \zeta_a(0) = \zeta_a(1) = 0\,, \qquad (6.52)
\end{aligned}$$

we find

$$\begin{aligned}
\langle \lambda^a(t) y^i(s) \rangle_0 &= i\hbar T^{ai}\,, \\
\langle \bar{x}_j(t) y^i(s) \rangle_0 &= i\hbar \delta_j^i [t - \vartheta(t-s)]\,, \\
\langle \bar{\eta}^b(t) \dot{\zeta}_a(s) \rangle_0 &= i\hbar \delta_a^b [t - \vartheta(t-s)]\,. \qquad (6.53)
\end{aligned}$$

Here we use the following definition of ϑ-function:[7]

$$\vartheta(t) = \begin{cases} 1, & t > 0; \\ 0, & t \leq 0. \end{cases} \qquad (6.54)$$

The quasi-classical expansion for the quantum average of a boundary observable $F(x(1))$ is given by

$$\langle F \rangle = \int F e^{\frac{i}{\hbar}\mathcal{S}_{\mathrm{gf}}} = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} \int F (\mathcal{S}_{\mathrm{int}})^n e^{\frac{i}{\hbar}\mathcal{S}_0} \qquad (6.55)$$

[7]There is an unavoidable ambiguity in the definition of $\vartheta(0)$. The value $\vartheta(0)$ contributes to the path integral trough the tadpole diagrams involving $\langle \bar{x}_j(t) y^i(t) \rangle_0$ and $\langle \bar{\eta}^b(t) \dot{\zeta}_a(t) \rangle_0$. The advantage of our choice $\vartheta(0) = 0$ is that it leads to a covariant expression for the first quantum correction (6.56). A similar problem for the path-integral quantization of the Poisson sigma-model is discussed in ref. [15].

Using the Wick theorem for Gaussian integral we find the following expression for the first quantum correction to the classical average:

$$\langle F\rangle = F(x_0) + \frac{i\hbar}{2}\Big[\nabla_i(G^{ij}\partial_j F) - \nabla_i(G^{ij}\partial_j T_a)T^{ak}\partial_k F - U^b_{ab}T^{ai}\partial_i F\Big](x_0) + O(\hbar^2)\,. \quad (6.56)$$

Here ∇ is some connection on M. The symmetric matrix $G^{ij} \equiv V^i_a T^{aj}$ can be thought of as the Feynman propagator of boundary fields (cf. (4.26))

$$\langle y^i(t)y^j(s)\rangle = i\hbar G^{ij}(x_0) + O(\hbar^2) \quad (6.57)$$

The expression for the first quantum correction is explicitly invariant under the general coordinate transformations on M and, as one can easily check, does not depend on the choice of connection ∇.

6.3 First-order theories

Let N be a smooth manifold equipped with a vector field $h = h^i(x)\partial_i$. The integral trajectories of h are defined by the system of first-order ODEs

$$T^i(x(t)) \equiv \dot{x}^i(t) - h^i(x(t)) = 0\,, \quad (6.58)$$

where the overdot stands for the derivative in time $t \in [t_1, t_2]$. To identify these equations with the general eqs. (2.3) of section 2 one should combine the discrete index i and the continuous evolution parameter t into the one superindex $a = (i, t)$. Then the space of histories M is the space of all smooth trajectories on N. Given eqs. (6.58), we look for a Lagrange anchor of the form

$$V^{ij}(t,s) = \alpha^{ij}(x(t))\delta(t-s)\,, \quad (6.59)$$

where $\alpha = \alpha^{ij}\partial_i \otimes \partial_j$ is a contravariant tensor on N. Substituting the ansatz (6.59) into the integrability condition (2.7) we get a set of necessary and sufficient conditions for the anchor V to be compatible with equations of motion. These conditions read

$$\alpha^{ij} = -\alpha^{ji}\,, \qquad [\alpha,\alpha] = 0\,, \qquad [h,\alpha] = 0\,. \quad (6.60)$$

Here the square brackets denote the Schouten commutator of multivector fields. Rels. (6.60) just say that α is a Poisson bivector on N, and the vector field h is a differentiation of the corresponding Poisson algebra.

We thus see that the Poisson structure is a particular example of the Lagrange one. One recovers it by looking for a local, purely algebraic anchor (6.59) for the first-order ODEs (6.58). When the Poisson bivector α is nondegenerate so is the anchor V. In that case the equations (6.58) appear to be hamiltonian and can be derived from a (local) action functional. Meanwhile, for a degenerate anchor V no such action can exist even if the equations (6.58) are hamiltonian.[8]

[8] The last relation in (6.60) is automatically satisfied if h is a (locally) hamiltonian vector field, i.e. $h = \rho_i\alpha^{ij}\partial_j$ with $\rho = \rho_i dx^i$ being a closed 1-form on N. For degenerate Poisson bivector the differentiation h, can be not necessarily hamiltonian even locally, so the equations (6.58) are more general than the Hamilton ones.

In accordance with the definitions of section 3, rels. (6.58), (6.59), (6.60) define a regular Lagrange structure of type $(0,0)$. The corresponding BRST charge, being constructed by the general method of section 4, reads

$$\Omega = \int_{t_1}^{t_2} dt \left(\bar{\eta}_i(\dot{x}^i - h^i + \alpha^{ij}\bar{x}_j) + \frac{1}{2}\bar{\eta}_i\bar{\eta}_j\nabla_k\alpha^{ij}\eta^k \right), \tag{6.61}$$

Here $\nabla = \partial + \Gamma$ is an arbitrary symmetric connection N. If we set $\bar{\eta}_i(t_1) = \bar{\eta}_i(t_2) = 0$, then the BRST charge meets the master equation $\{\Omega, \Omega\} = 0$ with respect to the following Poisson bracket:

$$\begin{aligned}
\{\bar{\eta}^i(t), \eta_j(s)\} &= \delta^i_j \delta(t-s)\,, \qquad \{\bar{x}_i(t), \eta_j(s)\} = \Gamma^k_{ij}\eta_k\delta(t-s)\,,\\
\{\bar{x}_i(t), x^j(s)\} &= \delta^j_i \delta(t-s)\,, \qquad \{\bar{x}_i(t), \bar{\eta}^j(s)\} = -\Gamma^j_{ik}\bar{\eta}^k\delta(t-s)\,,\\
\{\bar{x}_i(t), \bar{x}_j(s)\} &= R^n_{ijk}\bar{\eta}_n\eta^k\delta(t-s)\,,
\end{aligned} \tag{6.62}$$

$R^n_{ijk}(x)$ being the curvature tensor of ∇. Substituting the BRST charge (6.61) and the symplectic potential for bracket (6.62) into the general formula (5.3), one can get the topological sigma model whose dynamics is equivalent to the original first order dynamics (6.58). Having this topological sigma-model, one can compute the average values (transition amplitudes) for physical observables by the formula (5.11), even though the original equations (6.58) are not hamiltonian.

Notice that the expression (6.61) would reproduce the BRST charge of the Poisson sigma-model if h was set to zero and $\nabla = \partial$. This might be viewed as a possible answer to the question [15] about the way of incorporating the hamiltonian into the Poisson sigma-model. The BRST charge (6.61), containing the covariant derivative of the Poisson bivector and nilpotent with respect to the non-canonical Poisson bracket (6.62) probably answers to one more question [29], about the way of incorporating connection in the BFV-BRST quantization of Poisson sigma-model to make it explicitly covariant in the target space. Notice that the "covariantization" of BFV scheme was given in [22] for general first-class constraint systems. The above BRST-BFV formulation for the topological sigma-model can be viewed as a particular case of the covariant formalism of the paper [22].

This example allows further extension: the first-order equations (6.58) can be complemented by the constraints reducing the dynamics to a submanifold in N. Also the (sub)manifold can be factorized by a gauge symmetry. In this case, the ansatz (6.59) for the Lagrange anchor would result in the requirement for α to satisfy the Jacobi identity modulo constraints and gauge generators. Dynamics of this type have been recently studied in [3], where the deformation quantization method was extended to such systems. Similar manifolds were also studied in [9]. The method of the present paper allows us to define the transition amplitudes for such systems using the gauged version of the sigma-model defined by the BRST charge (6.61).

6.4 Maxwell electrodynamics in the first-order formalism

In this section, we exemplify the general quantization method by the model of Maxwell electrodynamics in first-order formalism. This is a simple example, which demonstrates many characteristic features of more complicated non-Lagangian field-theoretical models.

In the first-order formalism, the electromagnetic field can be described by the strength tensor considered as an independent field, i.e. without use of the electromagnetic potential. The equations of motion for the strength tensor are not lagrangian. These equations are dependent, i.e. there are Noether identities (2.8) among them, but there are no gauge transformations for the fields. The dynamics bundle (see remark 4 of section 3.2) is different from the cotangent bundle, even by dimension. So this model allows us to exemplify how the quantization method can handle with all these features that are impossible in the lagrangian dynamics. On the other hand, as the Maxwell electrodynamics admits alternative lagrangian formulation involving the electromagnetic potential, one can check that the quantization performed by our method gives the same results as in the standard lagrangian formalism.

The Maxwell equations for strength tensor $F_{\mu\nu}(x)$ read

$$T_{1\mu}(x) \equiv \partial^\nu F_{\mu\nu}(x) - J_\mu(x) = 0\,, \qquad T_{2\mu}(x) \equiv \partial^\nu \tilde{F}_{\mu\nu}(x) = 0\,, \tag{6.63}$$

where $J_\mu(x)$ is a conserved electric current (considered as an external source), and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{6.64}$$

denotes the dual strength tensor. All indices are risen and lowered by Minkowski metric in $\mathbb{R}^{1,3}$. Due to the antisymmetry of the strength tensor, eqs. (6.63) are linearly dependent,

$$\partial^\mu T_{z\mu} \equiv 0\,, \qquad z = 1, 2\,. \tag{6.65}$$

To make contact with general notation of the paper introduced in section 2, and to identify general relations (2.3), (2.8) with (6.63,) (6.65) in the Maxwell theory, one should collect the discrete and continuous indices into the following superindices: $a = (z, \mu, x)$, $A = (z, x)$ and $i = ([\mu\nu], x)$.

Consider the Lagrange anchor $V_a = (V_{1\mu}, V_{2\mu})$, where the Poincaré covariant vector fields

$$V_{1\mu} = 0\,, \qquad V_{2\mu} = \int d^4y \partial^\nu \delta^4(x-y)\frac{\delta}{\delta \tilde{F}^{\mu\nu}(y)}\,. \tag{6.66}$$

form an abelian distribution. Since

$$V_{2\mu}T_{2\nu} = \frac{1}{2}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\delta^4(x-y)\,, \quad V_{2\mu}T_{1\nu} = 0\,, \tag{6.67}$$

the integrability condition (2.7) is obviously satisfied. The anchor is regular but not complete (see section 3.3). The physical explanation for this incompleteness is that the second equation in (6.63), expressing the fact of closedness of 2-form $F_{\mu\nu}(x)dx^\mu \wedge dx^\nu$, is considered as non-lagrangian in the sense of proposition 1. Hence, no quantum fluctuations violate the condition $dF = 0$, and it could be (locally) resolved in terms of electromagnetic potentials $A = A_\mu(x)dx^\mu$ both at classical and quantum levels. (Recall that according to rel. (6.50) the non-lagrangian equations of motion enter the path integral as arguments of δ-function thereof suppressing any quantum fluctuations of “non-lagrangian” degrees of freedom).

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According to the classification of section 3.2, rels. (6.63), (6.65) and (6.66) describe a regular Lagrange structure of type (0,1). The corresponding BRST charge is given by

$$\Omega = \int d^4x \left(\bar{\eta}^{2\mu}[\partial^\nu \tilde{F}_{\mu\nu} + \partial^\nu \tilde{\bar{F}}_{\mu\nu}] + \bar{\eta}^{1\mu}[\partial^\nu F_{\mu\nu} - J_\mu] + \bar{\xi}^z \partial^\mu \eta_{z\mu}\right), \tag{6.68}$$

$\bar{F}_{\mu\nu}$ being the momenta canonically conjugated to the fields $F^{\mu\nu}$. Substituting the BRST charge to the general expression (5.3) for the sigma-model action yields [9]

$$\mathcal{S} = \int_0^1 dt d\theta \left[\Omega + \int d^4x (\bar{F}_{\mu\nu} D F^{\mu\nu} + \bar{\eta}^{z\mu} D\eta_{z\mu} + \bar{\xi}^z D\xi_z)\right]. \tag{6.69}$$

Integrating of θ, we get

$$\begin{aligned}\mathcal{S} = \int_0^1 dt \int d^4x \Big(&\bar{F}_{\mu\nu}\dot{F}^{\mu\nu} - \bar{\eta}^{z\mu}\dot{\eta}_{z\mu} + \bar{\xi}^z\dot{\xi}_z + \overset{*}{\bar{\eta}}{}^{2\mu}[\partial^\nu \tilde{F}_{\mu\nu} + \partial^\nu \tilde{\bar{F}}_{\mu\nu}] + \overset{*}{\bar{\eta}}{}^{1\mu}[\partial^\nu F_{\mu\nu} - J_\mu] - \\ &- \bar{\eta}^{2\mu}\partial^\nu \overset{*}{\tilde{\bar{F}}}{}_{\mu\nu} - \bar{\eta}^{2\mu}\partial^\nu \overset{*}{\tilde{F}}{}_{\mu\nu} - \bar{\eta}^{1\mu}\partial^\nu \overset{*}{F}{}_{\mu\nu} + \bar{\xi}^z\partial^\mu \overset{*}{\eta}_{z\mu} + \overset{*}{\bar{\xi}}{}^z \partial^\mu \eta_{z\mu}\Big). \end{aligned} \tag{6.70}$$

The action is invariant under the following gauge transformation:

$$\begin{aligned} &\delta F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\rho \bar{\varepsilon}^2_\sigma\,, \qquad \delta\bar{F}_{\mu\nu} = \frac{1}{2}\partial_{[\mu}\bar{\varepsilon}^1_{\nu]} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho \bar{\varepsilon}^{2\sigma}\,, \qquad \delta \overset{*}{\bar{\eta}}{}^{z\mu} = -\dot{\bar{\varepsilon}}^{z\mu} + \partial^\mu \overset{*}{\bar{\varepsilon}}{}^z\,, \\ &\delta\bar{\eta}^{z\mu} = -\partial^\mu \bar{\varepsilon}^z\,, \qquad \delta\overset{*}{\bar{\xi}}{}^z = -\dot{\bar{\varepsilon}}^z\,, \qquad \delta\bar{\xi}^z = 0\,. \end{aligned} \tag{6.71}$$

(Here we write out only the transformation formulas for fields.)

From the general viewpoint, (6.70) is a minimal master action of a gauge theory with linearly dependent gauge-algebra generators. The fields $\bar{\eta}$'s and $\overset{*}{\bar{\xi}}$'s are ghosts associated to the reducible gauge symmetry, and the fields $\bar{\eta}$'s play the role of ghosts-for-ghosts.

The gauge fixing procedure for the theory at hands is standard [2]. We introduce the "non-minimal sector of variables" constituted by the trivial field-antifield pairs (b's, $\overset{*}{b}$'s) and (π's, $\overset{*}{\pi}$'s). The spectrum, parity and ghost number of these fields are completely determined by the form of gauge transformations (6.71) (see [2] for details). An appropriate gauge-fixing fermion is chosen as

$$\Psi = \int_0^1 dt \int d^4x (b_{z\mu}\dot{\lambda}^{z\mu} + \dot{b}_z \partial_\mu \lambda^{z\mu} + b_{1z}\partial_\mu \bar{\eta}^{z\mu} + b_1^{1z}\partial^\mu \dot{b}_{z\mu})\,, \qquad \mathrm{gh}(\Psi) = -1\,, \tag{6.72}$$

where we set $\lambda^{z\mu} \equiv \overset{*}{\bar{\eta}}{}^{z\mu}$. The gauge-fixed action is obtained by excluding the antifields from the non-minimal master action

$$S + \int_0^1 dt \int d^4x \left(\pi_{z\mu}\overset{*}{b}{}^{z\mu} + \pi_z \overset{*}{b}{}^z + \pi_{1z}\overset{*}{b}{}^z_1 + \pi_1^{1z}\overset{*}{b}{}^1_{1z}\right). \tag{6.73}$$

[9] Both the dynamics bundle and the Noether identity bundle are assigned with flat connection.

using the equation $\phi^*_i = \delta\Psi/\delta\phi^i$, where ϕ^i and ϕ^*_i collectively denote all the fields and antifields respectively. The result is

$$\begin{aligned}\mathcal{S}_{\rm gf} = \int d^4x \Bigg\{ \int_0^1 dt \Big[& \bar{F}_{\mu\nu}\dot{F}^{\mu\nu} + \bar{\eta}^{z\mu}(\ddot{b}_{z\mu} + \partial_\mu \ddot{b}_z) + \lambda^{2\mu}(\partial^\nu \tilde{F}_{\mu\nu} + \partial^\nu \tilde{\bar{F}}_{\mu\nu}) + \lambda^{1\mu}(\partial^\nu F_{\mu\nu} - J_\mu) + \\ & + b_{1z}\Box\bar{\xi}^z - (\partial^\mu \dot{b}_{z\mu} + \Box\dot{b}_z)\overset{*}{\bar{\xi}}{}^z + \pi_{z\mu}(\dot{\lambda}^{z\mu} + \partial^\mu \dot{b}_1^{1z}) - \\ & - \pi_z\partial_\mu\dot{\lambda}^{z\mu} + \pi_{1z}\partial_\mu\bar{\eta}^{z\mu} + \pi_1^{1z}\partial^\mu\dot{b}_{z\mu}\Big] + \\ & + \left(\dot{\bar{\eta}}^{z\mu}b_{z\mu} + b_{z\mu}\partial^\mu\overset{*}{\bar{\xi}}{}^z + \pi_{z\mu}\partial^\mu b_1^{1z} + \pi_z\partial_\mu\lambda^{z\mu}\right)\Bigg|_0^1 \Bigg\}. \end{aligned} \tag{6.74}$$

We impose the following boundary conditions at $t = 0, 1$:

$$\bar{\eta}^{z\mu} = 0\,, \qquad \bar{\xi}^z = 0\,, \qquad \pi_1^{1z} = 0\,, \qquad b_{z\mu} = -\partial_\mu b_z\,, \qquad b_1^{1z} = 0\,. \tag{6.75}$$

The physical observables are just arbitrary functionals of the strength tensor $F_{\mu\nu}(x)$ and the quantum average $\langle\mathcal{O}\rangle$ of an observable $\mathcal{O}(F_{\mu\nu})$ is given by the general expression (6.45). As with an arbitrary free theory, the ghost dynamics is completely decoupled from the dynamics of physical fields. Integrating over the interior values of ghost fields we arrive at the path integral with action

$$\begin{aligned}\mathcal{S}_{\rm gf} = \int d^4x \Bigg\{ \int_0^1 dt \, \Big[& \bar{F}_{\mu\nu}\dot{F}^{\mu\nu} + A^{2\mu}(\partial^\nu \tilde{F}_{\mu\nu} + \partial^\nu \tilde{\bar{F}}_{\mu\nu}) + A^{1\mu}(\partial^\nu F_{\mu\nu} - J_\mu)\Big] + \\ & + \left(b_z\Box\overset{*}{\bar{\xi}}{}^z + \pi_z\partial_\mu A^{z\mu}\right)\Bigg|_0^1 \Bigg\}, \end{aligned} \tag{6.76}$$

where $A^{z\mu}(x) \equiv \lambda^{z\mu}(x,0)$. (One can check that all the determinants resulting from the integration are cancelled out.) Then, integrating of the remaining variables $F_{\mu\nu}$, $\bar{F}_{\mu\nu}$, $A^{z\mu}$, b_z, $\overset{*}{\bar{\xi}}{}^1$ and π_z, we arrive at the final result

$$\langle\mathcal{O}\rangle = (\mathrm{const}) \int \mathcal{D}A\mathcal{D}F\mathcal{D}\pi\mathcal{D}c\mathcal{D}\bar{c}\,\mathcal{O}(F_{\mu\nu})e^{\frac{i}{\hbar}\tilde{S}_{\rm gf}}\,. \tag{6.77}$$

Here we introduced the following notation:

$$\tilde{S}_{\rm gf} = \int d^4x\Big(F^{\mu\nu}[\partial_\mu A_\nu + F_{\mu\nu}] + \pi\partial^\mu A_\mu + \partial_\mu c\partial^\mu \bar{c}\Big), \tag{6.78}$$

$$A_\mu = A_{1\mu}(x)\,, \qquad F_{\mu\nu} = F_{\mu\nu}(x,1)\,, \qquad \pi = \pi_1(x,1)\,, \qquad c = \overset{*}{\bar{\xi}}{}^1(x,1)\,, \qquad \bar{c} = b_1(x,1)\,. \tag{6.79}$$

As is seen the functional (6.78) is nothing but the gauge-fixed action of Maxwell's electrodynamics in the first-order formalism: The boundary values of the Lagrange multipliers $\lambda_{1\mu}$ are identified with electromagnetic potentials A_μ, whereas the boundary values of ghost-for-ghost $\overset{*}{\bar{\xi}}{}^1$ and b_1 play the role of the Faddeev-Popov ghosts associated with the gauge invariance $A_\mu \to A_\mu + \partial_\mu\varphi$ and the Lorentz gauge $\partial^\mu A_\mu$.

Above, one can see how the non-lagrangian Maxwell's equations for the strength tensor (6.63) are quantized strictly following the general prescription of the paper, through quantizing equivalent topological field theory in five dimensions (6.70). After integrating of the fields in the bulk (that can be done explicitly in this case) the result reduces to the standard expression (6.77) for the quantum average given by the Faddeev-Popov quantization of electrodynamics in terms of electromagnetic potential.

7. Concluding remarks

In this paper we have proposed a method of converting any not necessarily lagrangian dynamics into equivalent lagrangian topological field theory, that allows us to path-integral quantize the dynamical system. The Lagrange structure described in sections 2 and 3 is a key pre-requisite of this quantization method. Given the equations of motion (2.3), they completely define their gauge symmetries (2.9) and the Noether identities (2.8), that are the basic ingredients for constructing the Lagrange structure. The Lagrange anchor (2.7) is the last input the Lagrange structure needs to be defined. This ingredient can not be uniformly found from the equations of motion, as the compatibility conditions (2.7) are not so restrictive to determine the anchor in a unique way. This is much similar to the Poisson structure which is not uniquely defined by given first order equations of motion (As is seen from the example of section 6.3, this is more than a general analogy). Different Lagrange anchors can result in different quantization for the same classical theory. The trivial anchor $V = 0$, which is compatible with any equations of motion, results in a trivial quantization in the sense that the quantum average of any physical observable would coincide with its classical value. Choosing a degenerate Lagrange anchor of constant rank, one implicitly separates the degrees of freedom which are quantized from those which are not. This is a choice which can have physical and/or geometrical motivation, but it can not be justified just by the form of the equations of motion. Even in the simplest case with equations of motion following from the action with no gauge symmetry, one can choose, in principle, not a unit anchor (that would result in the standard Feynman path integral) but a degenerate one that would assign no quantum corrections to some degrees of freedom. And this latter choice might have reasonable physical motivation in some cases, e.g. when some "part" of dynamics is to be considered as an effective theory emulating classical background for the other part that demonstrates a quantum behavior.

So, the general conclusion is that the Lagrange structure which is behind the path integral quantization, requires more physical/geometrical data about the system than it is contained in the classical equations of motion. This is not surprising: as quantum theory gives a more detailed description of the model, it has to require more inputs.

There are many physically interesting models, like self-dual Yang-Mills theory, Vasiliev's higher-spin fields [20], etc., having no lagrangians. These models can have, however, nontrivial Lagrange structures that could allow to quantize these non-lagrangian theories. As the equations of motion are not sufficient, in general, to uniquely define the Lagrange anchor, one has to identify in these models some other appropriate structures that could serve as building blocks for constructing the anchor. There are various ideas

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that can be helpful for making such an identification in local field theories. For example, not so many explicitly relativistic covariant finite order differential operators can be found for a given field, so even the most general ansatz for a local explicitly covariant Lagrange anchor can be amenable to study in many cases. As is seen from the example of section 6.4, there are quite few first order operators for the abelian vector field that can be tested for this role. If the space of trajectories M admits any symmetric contravariant second rank tensor G^{ij} (degenerate or not, no matter) even not related anyhow with the equations of motion, it can be taken as a propagator of section 4.4, that would define the Lagrange anchor in the form $V_a^i = \partial_j T_a G^{ji}$. Even the requirement of symmetry can be relaxed for G^{ij} to hold on shell only. If other geometric data are naturally assigned to the system, other schemes can be invented, perhaps, to convert these data into the Lagrange anchor. In this paper we just propose the method to path integral quantize dynamics, given equations of motion and Lagrange anchor.

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